Math 1232: Single-Variable Calculus 2

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Recitation 9

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Problem 1 (Geometric Series)**.** Compute:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{3^k}$

(b) $\displaystyle\sum_{k=2}^{\infty} \frac{(-5)^{k+2}}{2^{3k}}$

(c) $\frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \frac{5}{16} + \dots$

(d) $\frac{-2}{3} + \frac{8}{9} + \frac{-32}{27} + \dots$

(e) $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \dots$

Solution:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{3^k} = \frac{2/3}{1-2/3} = 2.$

(b) $\displaystyle\sum_{k=2}^{\infty} \frac{(-5)^{k+2}}{2^{3k}} = \frac{625/64}{1+5/8} = \frac{625/64}{13/8} = \frac{625}{104}.$

(c) $\frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \frac{5}{16} + \dots = \sum_{k=1}^{\infty} \frac{5}{2^k} = \frac{5/2}{1-1/2} = 5.$

(d) $\frac{-2}{3} + \frac{8}{9} + \frac{-32}{27} + \dots = \sum_{k=1}^{\infty} \frac{-2}{3}\frac{4^{k-1}}{3^{k-1}}$ and since the ratio $r = \frac{4}{3} > 1$ this series diverges.

(e) $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \dots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{3^k} = \frac{1/3}{1+1/3} = \frac{1/3}{4/3} = \frac{1}{4}.$

Problem 2 (Infinite Decimals)**.** We want to find a rational representation of the infinite decimal $0.\overline{47}$. That is, we want to write $0.\overline{47} = \frac{p}{q}$ for integers p, q.

(a) First, what happens if we multiply $0.\overline{47}$ by 100?

(b) Using part (a), what can you tell about $(99) \cdot 0.\overline{47}$?

(c) Give a rational representation of $0.\overline{47}$.

(d) Now let's take a different approach. Write $0.\overline{47}$ as an infinite series.

(e) What kind of series is this? Can you use that fact to find a rational representation of $0.\overline{47}$?

(f) Now use the same logic to find a rational representation of $2.\overline{63}$.

Solution:

(a) $0.\overline{47} \cdot 100 = 47.\overline{47}$.

(b)

$$\begin{aligned} 99 \cdot 0.\overline{47} &= 100 \cdot 0.\overline{47} - 0.\overline{47} \\ &= 47.\overline{47} - 0.\overline{47} = 47 \end{aligned}$$

(c) Thus $0.\overline{47} = \frac{47}{99}$.

(d) We can write $0.\overline{47} = \sum_{k=1}^{\infty} 47 \cdot 100^{-k}$.

(e) This is a geometric series with $a = \frac{47}{100}$ and $r = \frac{1}{100}$. Thus

$$0.\overline{47} = \sum_{k=1}^{\infty} 47 \cdot 100^{-k} = \frac{47/100}{1 - 1/100} = \frac{47/100}{99/100} = \frac{47}{99}.$$

(f) We can ignore the 2 until later. We can see

$$\begin{aligned} 0.\overline{63} &= \sum_{k=1}^{\infty} 63 \cdot 100^{-k} \\ &= \frac{63/100}{1 - 1/100} = \frac{63/100}{99/100} \\ &= \frac{63}{99} \\ 2.\overline{63} &= 2 + \frac{63}{99} = \frac{261}{99}. \end{aligned}$$

Problem 3. For each of the following series, write a careful argument showing either that it converges or that it diverges. Think about exactly what test you want to use and why.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{5n^3-2}{3n^5-n}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{n^3\ln(n)+1}{n^4-7}$.

Solution:

(a) This is a pile of polynomials, so it'll be simplest to use the limit comparison test. It looks like $\frac{n^3}{n^5} = \frac{1}{n^2}$, so we compute

$$\lim_{n\to\infty} \frac{\frac{5n^3-2}{3n^5-n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{5n^5-2n^3}{3n^5-n} = 5/3.$$

This is a real finite non-zero limit. Then since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges as a p-series, our original series converges by the Limit Comparison Test.

(b) This doesn't look at all geometric, but also isn't just polynomials, so we hope the regular comparison test works. This looks kinda like $\frac{n^3}{n^4} = \frac{1}{n}$. And in fact we see $n^3\ln(n)+1 > n^3\ln(n) > n^3$ as long as $n > 2$, and $n^4 - 7 < n^4$. So we have

$$\frac{n^3\ln(n)+1}{n^4-7} > \frac{n^3}{n^4} = \frac{1}{n}.$$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by the p-series test, we know that our original series diverges by the comparison test.

Problem 4 (Bonus)**.** Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n^2+e^n)}{n^2}$ converge or diverge?

Solution: We know that $0 \le \sin^2(n^2+e^n) \le 1$, so

We have that $\frac{\sin^2(n^2+e^n)}{n^2-n} \le \frac{1}{n^2}$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the $p-$ series test. So by the comparison test the series $\sum_{n=1}^{\infty} \frac{\sin^2(n^2+e^n)}{n^2-n}$ converges.