Math 1231 Midterm 2 Solutions

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Problem 1 (M3). (a) The function $f(x) = \frac{x^2 - 4x + 8}{2x - 1}$ has absolute extrema either on the interval $[-3, 0]$ or on the interval $[0, 3]$. Pick one of those intervals, explain why f has extrema on that interval, and find the absolute extrema.

Solution: f is continuous on the closed interval $[-3, 0]$, so it must have an absolute maximum and an absolute minimum on that interval.

$$\begin{aligned} f'(x) &= \frac{(2x-4)(2x-1) - 2(x^2 - 4x + 8)}{(2x-1)^2} \\ &= \frac{4x^2 - 8x - 2x + 4 - 2x^2 + 8x - 16}{(2x-1)^2} \\ &= \frac{2x^2 - 2x - 12}{(2x-1)^2} \\ &= \frac{2(x-3)(x+2)}{(2x-1)^2} \end{aligned}$$

so there are critical points at $3, -2, 1/2$. The only one we care about is -2. So we compute

$$\begin{aligned} f(-3) &= \frac{29}{-7} \\ f(-2) &= \frac{20}{-5} = -4 \\ f(0) &= \frac{8}{-1} = -8. \end{aligned}$$

so the minimum is -8 at 0, and the maximum is -4 at -2.

(b) Find and classify the critical points of $f(x) = x^3 + 2x^2 - 4x + 5$.

Solution: We compute $f'(x) = 3x^2 + 4x - 4 = (3x - 2)(x + 2)$ so this function has critical points at $2/3$ and -2.

From here we have two choices. We can make a chart and use the first derivative test:

	$3x-2$	$x+2$	$f'(x)$
$x < -2$	$-$	$-$	$+$
$-2 < x < 2/3$	$-$	$+$	$-$
$2/3 < x$	$+$	$+$	$+$

so f has a relative maximum at $x = -2$ and a relative minimum at $x = 2/3$.

Or we can use the second derivative test: $f''(x) = 6x + 4$. We have $f''(-2) = -8 < 0$ so f has a maximum at $x = -2$, and $f''(2/3) = 8 > 0$ so f has a minimum at $x = 2/3$.

Problem 2 (S4)**.** The force a magnet exerts on a piece of iron depends on the distance between the magnet and the metal. Let $F(d) = \frac{2}{d^2}$ give the force exerted by the magnet in Newtons, where d is the distance between them in meters.

(i) What are the units of $F'(d)$? What does it $F'(d)$ represent physically? What would it mean if $F'(d)$ is big?

(ii) Calculate $F'(2)$. What does this tell you physically? What physical observation could you make to check your calculation?

Solution:

(i) The derivative is the rate at which the amount of force changes as you change the distance between the magnet and the iron; its units are Netwons per meter. If $F'(d)$ is big, that means that moving the magnet a little bit will change the force on it by a lot.

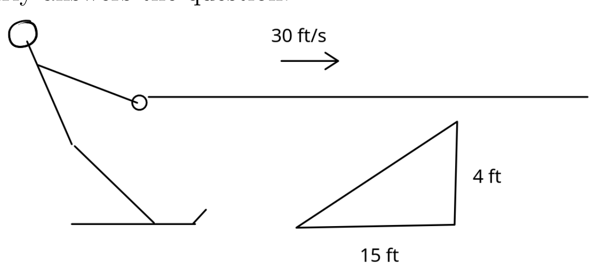
(ii) $F'(d) = \frac{-4}{d^3}$ so $F'(3) = \frac{-4}{8} = -1/2$. This means that moving the iron another meter away from the magnet should reduce the force by about half a Newton.

Problem 3 (S5)**.** Find a formula for y' in terms of x and y if $y\cos(xy) = 3y^2 + x$.

Solution:

$$y'\cos(xy) - y\sin(xy)(y + xy') = 6yy' + 1$$
$$y'\cos(xy) - xyy'\sin(xy) - 6yy' = 1 - y^2\sin(xy)$$
$$y' = \frac{1 - y^2\sin(xy)}{\cos(xy) - xy\sin(xy) - 6y}.$$

Problem 4 (S6)**.** A waterskier is moving horizontally at 30ft/s and rides up a ramp that is 15ft long and 4ft tall. How fast is she rising as she leaves the ramp? Answer in a complete sentence that directly and clearly answers the question.



Solution: We want to know how quickly the waterskier is moving upwards, and we know how fast she's moving horizontally. She's moving in a triangular direction based on the ramp, so we can relate her vertical and horizontal position with the similar triangles formula.

By similar triangles, we know that $\frac{h}{w} = \frac{4}{15}$; solving gives us $15h = 4w$. Taking a derivative gives $15h' = 4w'$. We know that $w' = 30\text{ft/s}$, so we get

$$15h' = 4 \cdot 30\text{ft/s}$$
$$h' = 8\text{ft/s}.$$

Thus the waterskier is rising at eight feet per second when she leaves the ramp.

Problem 5 (S7)**.** Let $f(x) = \sqrt[3]{x^2 - 2x} = \sqrt[3]{x(x-2)}$. We compute that

$$f'(x) = \frac{2(x-1)}{3x^{2/3} \cdot (x-2)^{2/3}}$$
$$f''(x) = \frac{-2(x^2 - 2x + 4)}{9(x-2)^{5/3} \cdot x^{5/3}}.$$

Sketch a graph of f. Your answer should *explicitly* discuss the domain, roots, limits at infinity, critical points and values, intervals of increase and decrease, and potential points of inflection, and concavity.

Solution: The function is defined everywhere. We see there are roots at $x = 0, 2$, and $\lim_{x\to\pm\infty} f(x) = +\infty$.

We see that $f'(x)$ is undefined at $x = 0, 2$, and is zero when $x = 1$. So our critical points occur at $0, 1, 2$. We calculate $f(0) = f(2) = 0$, and $f(1) = \sqrt[3]{-1} = -1$. By making a chart, we get

	$2(x-1)$	$x^{-2/3}/3$	$(x-2)^{-2/3}$	f'
$x < 0$	$-$	$+$	$+$	$-$
$0 < x < 1$	$-$	$+$	$+$	$-$
$1 < x < 2$	$+$	$+$	$+$	$+$
$2 < x$	$+$	$+$	$+$	$+$

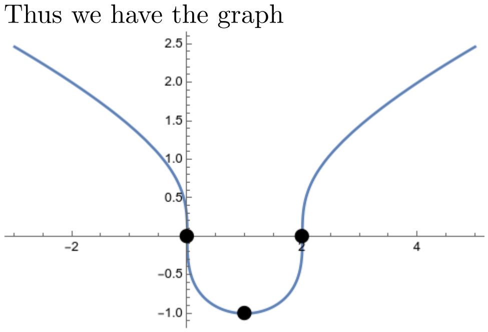
so f is decreasing on $(-\infty, 1)$ and it's increasing on $(1, +\infty)$.

The second derivative is undefined at $0, 2$ and is zero when $x^2 - 2x + 4 = 0$, which never happens. We still have $f(0) = f(2) = 0$. We can again make a chart:

	$-2(x^2 - 2x + 4)$	$(x-2)^{-5/3}/9$	$x^{-5/3}$	f''
$x < 0$	$-$	$-$	$-$	$-$
$0 < x < 2$	$-$	$-$	$+$	$+$
$2 < x$	$-$	$+$	$+$	$-$

so the function is concave up for $0 < x < 2$, and it's concave down for $x < 0$ and $x > 2$.

Thus we have the graph



Problem 6 (S8)**.** Find the point on the line $y = 2x + 5$ that is closest to the origin.

Solution: Our objective function is $D = \sqrt{x^2 + y^2}$, and our constraint is that $y = 2x + 3$. So we have

$$\begin{aligned} D(x) &= \sqrt{x^2 + (2x+5)^2} = \sqrt{x^2 + 4x^2 + 20x + 25} \\ &= \sqrt{5x^2 + 20x + 25} \\ D'(x) &= \frac{10x + 20}{2\sqrt{5x^2 + 20x + 25}} \end{aligned}$$

has a critical point only at $x = -2$, which gives us the point $(-2, 1)$.

To check this is really a minimum, we can't really use the EVT since x can be infinitely big or small on the line. However, we can observe that the denominator of $D'(x)$ is always positive, but the numerator is positive for $x > -2$ and negative for $x < -2$, which makes it a minimum. Or if we really want to, we can compute the second derivative

$$\begin{aligned}
D''(x) &= \frac{20\sqrt{5x^2+20x+25} - (10x+20)\frac{10x+20}{\sqrt{5x^2+20x+25}}}{4(5x^2+20x+25)} \\
&= \frac{20(5x^2+20x+25) - (10x+20)^2}{4(5x^2+20x+25)^{3/2}} \\
&= \frac{100x^2+400x+500-100x^2-400x-400}{4(5x^2+20x+25)^{3/2}} \\
&= \frac{100}{4(5x^2+20x+25)^{3/2}}
\end{aligned}$$

which is always positive. So the distance function is concave up, so we have a minimum.