# 4 Interlude: Approximation

This section is a bit of an interlude; it'll be a short bridge between section 3 on optimization, and section 5 on integration.

In this section we want to talk a bit more about the idea of approximation. We introduced this in section 4, when we talked about continuous approximation: if $x \approx a$, we can estimate $f(x) \approx f(a)$. We refined this a bit in section 2.1 and 2.6. The derivative allows us to estiamte that $f(x) \approx f(a) + f'(a)(x-a)$. But can we do even better?

## 4.1 Quadratic Approximation

In this class we've spent a lot of time on *linear approximation*: we can approximate a function with its tangent line, which is the linear function most similar to our starting function. This simplifies a lot of things, but is only an approximation.

$$f(x) \approx f(a) + f'(a)(x-a). \tag{2}$$

How good this approximation is depends on two things. The first is the distance $|x-a|$; the approximation is better when your goal point $x$ is close to your starting point $a$. There are other techniques (like Fourier series) that don't have this limitation, but we won't discuss them in this course.

The other is the speed at which the derivative changes. If the derivative is constant, your function is just a line and the "approximation" is perfect. But the faster the derivative changes, the faster the function deviates from the line.

Thus we might try to get a better approximation using the second derivative, which tells us how quickly the derivative is changing. So how can we do this?

We're looking for some function $g(x)$ so that

$$f(x) \approx f(a) + f'(a)(x-a) + g(a)(x-a)^2.$$

(We want the linear approximation to be the same as (4), and we want the third derivative to be zero, so the only thing that can change at all is the degree two term). Taking derivatives of both sides gives us

$$\begin{aligned} f'(x) &\approx f'(a) + 2g(a)(x-a) \\ f''(x) &\approx 2g(a). \end{aligned}$$

Thus we set $g(a) = f''(x)/2$, and we get the equation

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2. \tag{3}$$

This is the *parabola* that best approximates our function near $a$.

**Example 4.1.** Let's again ask our old question: what is $\sqrt{5}$?

We use the function $f(x) = \sqrt{x}$ and we compute $f'(x) = \frac{1}{2\sqrt{x}}$ and $f''(x) = \frac{-1}{4\sqrt{x^3}}$. Then we have

$$\begin{aligned}
f'(4) &= \frac{1}{4} \\
f''(4) &= \frac{-1}{32} \\
f(x) &\approx f(4) + f'(4)(x-4) + \frac{f''(4)}{2}(x-4)^2 \\
&= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 \\
f(5) &\approx 2 + \frac{1}{4} - \frac{1}{64} = 2 + \frac{15}{64} \approx 2.23483.
\end{aligned}$$

We see we've slightly overcorrected: rather than being .014 too big, we're now .0012 too small.

**Example 4.2.** Compute the quadratic approximations of $\sin(x)$ and $\cos(x)$ centered at zero. Estimate $\sin(.01)$ and $\cos(.01)$? How does this relate to the Small Angle Approximation?

$$\begin{aligned}
\sin'(x) &= \cos(x) \\
\sin'(0) &= 1 \\
\sin''(x) &= -\sin(x) \\
\sin''(0) &= 0 \\
\sin(x) &\approx 0 + 1(x-0) + \frac{0}{2}(x-0)^2 = x \\
\sin(.01) &\approx .01.
\end{aligned}$$

Recall the small angle approximation told us that $\sin(x) \approx x$. Here we see that this is not just a linear approximation, but in fact also the quadratic approximation; the reason the small angle approximation worked so well is that it was correct ot second order.

$$\begin{aligned}
\cos'(x) &= -\sin(x) \\
\cos'(0) &= 0 \\
\cos''(x) &= -\cos(x) \\
\cos''(0) &= -1 \\
\cos(x) &\approx 1 + 0(x-0) - 1(x-0)^2 = 1 - \frac{x^2}{2} \\
\cos(.01) &\approx .99995.
\end{aligned}$$

**Example 4.3.** Let $g(x) = x^4 - 3x^3 + 4x^2 + 4x - 2$. Compute the quadratic approximations at $a = 0$ and at $a = -2$. Compare them to $g(x)$. Estimate $g(-1.97)$.

$$\begin{aligned}
g(0) &= -2 \\
g'(x) &= 4x^3 - 9x^2 + 8x + 4 \\
g'(0) &= 4 \\
g''(x) &= 12x^2 - 18x + 8 \\
g''(0) &= 8 \\
g(x) &\approx -2 + 4(x-0) + \frac{8}{x}x^2 = 4x^2 + 4x - 2.
\end{aligned}$$

Notice that this is just the lower-degree terms of our original polynomial!

$$\begin{aligned}
g(-2) &= 16 + 24 + 16 - 8 - 2 = 46 \\
g'(x) &= 4x^3 - 9x^2 + 8x + 4 \\
g'(-2) &= -32 - 24 - 16 + 4 = -80 \\
g''(x) &= 12x^2 - 18x + 8 \\
g''(-2) &= 48 + 36 + 8 = 92 \\
g(x) &\approx 46 - 80(x+2) + 46(x+2)^2 \\
f(-1.97) &\approx 46 - 80(.03) + 46(.009) = 43.6414.
\end{aligned}$$

However, if we take $h(x) = 4x^2 + 4x - 2$ and approximate near $-2$, we get

$$\begin{aligned}
h(-2) &= 6 \\
h'(x) &= 8x + 4 \\
h'(-2) &= -12 \\
h''(x) &= 8 \\
h''(-2) &= 8 \\
h(x) &\approx 6 - 12(x+2) + 4(x+2)^2 = 6 - 12x - 24 + 4x^2 + 16x + 16 \\
&= 4x^2 + 4x - 2 = h(x).
\end{aligned}$$

No matter where we center our approximation, the best quadratic approximation to our parabola is our original parabola.

**Example 4.4.** Now let's estimate $1.01^{25}$ using a quadratic approximation. We use the function $f(x) = (1+x)^{25}$, and center our approximation at $x = 0$. (Equivalently we could consider $g(x) = x^{25}$ and center our approximation at $x = 1$; the way I set it up is a bit more common).

We take $f'(x) = 25(1+x)^{24}$ so $f'(0) = 25$, and $f''(x) = 25 \cdot 24(1+x)^{23}$ so $f''(0) = 25 \cdot 24 = 600$. Then we have

$$\begin{aligned}
f(x) &\approx 1 + 25(x-0) + \frac{600}{2}(x-0)^2 = 1 + 25x + 300x^2 \\
1.01^{25} = f(.01) &\approx 1 + 25 \cdot .01 + 300 \cdot .0001 = 1 + .25 + .03 = 1.28.
\end{aligned}$$

Since $1.01^{25} \approx 1.28243$ this is pretty good.

What if we move a bit farther? If we want to estimate $1.04^{25}$ we get

$$1.04^{25} = f(.04) \approx 1 + 25 \cdot .04 + 300 \cdot .0016 = 1 + 1 + .48 = 2.48$$

while $1.04^{25} \approx 2.66584$. We've lost fidelity because our move away is bigger.

But while .4 is still much smaller than 1, this estimate is much worse than our estimate of $\sqrt{5}$ from earlier. Why is this much worse? Linear are bad for two reasons: either because $x$ and $a$ are far apart, or because the second derivative is large. Here we've taken care of the second derivative, but we haven't taken care of everything. Our quadratic approximations will be bad when the *third* derivative is large.

Finally, let's use this to estimate $2^{25}$. We get

$$2^{25} = f(1) \approx 1 + 25 \cdot 1 + 300 \cdot 1^2 = 326.$$

But $2^{25} = 33,554,432$, so this is very far off. We see here even more problems with the largeness of the higher derivatives.

### 4.1.1 Cubics and Beyond: Taylor Series

We can carry this logic further. We can work out that if we want to match the first *three* derivatives and get a cubic approximation, we get the formula

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3 \cdot 2}(x-a)^3.$$

More generally, we can get a degree-$n$ polynomial approximation, called the *Taylor polynomial of degree $n$*, with the formula

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3 \cdot 2}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If a function is infinitely differentiable, we can take an infinite sum here and get the *Taylor series*:

$$T_f(x,a) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots.$$

Most functions we're interested in are equal to their own Taylor series. (Not all functions are, though!) In particular, we can work out the following formulas:

$$\begin{aligned} \sin(x) &= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} + \ldots \\ \cos(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \ldots \\ e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots. \end{aligned}$$

Taylor series are extremely important in any sort of computational or advanced math, and you will talk about them a lot more if you take Calculus II.

However, in practice, just like we rarely use third or fourth derivatives, we rarely use approximations of degree higher than two. If the quadratic approximation doesn't pick up whatever you need to think about, we will do something else entirely.

## 4.2 Iterative Approximation: Newton's Method

In section 2.6 we saw that there were two things that make a linear approximation work better or worse. The first was the size of the second derivative; in section 4.1 we leveraged the second derivative to improve our approximations.

To keep things simple, we'll assume that we want to solve $f(x) = 0$. (If not, we can just subtract our number $y$ from both sides of the equation). If we know the value of $f$ and of $f'$ at a point $x_0$, then recall that by linear approximation we estimate that $f(x_1) =$

$f(x_0) + f'(x_0)(x_1 - x_0)$. Since we want $f(x_1) = 0$, we set $f(x_1) = 0$ and solve this equation for $x_1$, and get

$$x_1 = x_0 - (f(x_0)/f'(x_0)).$$

In many conditions, we will get the result that $x_1$ is closer to being a root of $f$ than $x_0$ is.

We can repeat this process to find $x_2$, $x_3$, etc., and ideally each will be a better estimate than the previous estimate was. A good rule of thumb for when to stop: if you want five decimal places of accuracy, you can stop when the $n$th step and the $n + 1$st step agree to five decimal places.

This method does have limitations. First, we have to start with a guess $x_1$ for our root $x$. Second, if $f'(x_1)$ is very close to zero, Newton's method will work poorly if it works at all, and we might have to pick a better guess. But it can be very useful for finding approximate solutions to equations.

**Example 4.5.** Let's approximate the square root of 5, one more time. First, we need to turn this into finding a solution to an equation. We want to solve the equation $x^2 = 5$, which we can rewrite as $f(x) = x^2 - 5 = 0$. We compute $f'(x) = 2x$.

We need to pick a starting estimate, which should probably be $x_0 = 2$. Then we have $f(x_0) = -1$, and $f'(x_0) = 4$. So we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{-1}{4} = 9/4 = 2.25.$$

You might notice that this is exactly what we got by doing a simple linear approximation. So what did we get from this new method? Now we can *iterate*.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 9/4 - \frac{81/16 - 5}{9/2} = 161/72 \approx 2.23611$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 161/72 - \frac{1/5184}{161/36} = \frac{51851}{23184} \approx 2.23607$$

Checking with a computer tells us that $\sqrt{5} \approx 2.23607$, so we're now correct to five decimal places.

**Example 4.6.** Let's find a solution to $x^3 - x = 1$. We need to write this as $f(x) = 0$, so let's take $f(x) = x^3 - x - 1$. Then we have $f'(x) = 3x^2 - 1$, and we can guess $x_0 = 1$ as a decent starting point, since $f(1) = -1$ is close to 0. Then we have

$$x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{2} = 3/2$$
$$x_2 = \frac{3}{2} - \frac{f(3/2)}{f'(3/2)} = \frac{3}{2} - \frac{27/8 - 3/2 - 1}{27/4 - 1} = 31/23 \approx 1.34783$$
$$x_3 = \frac{31}{23} - \frac{f(31/23)}{f'(31/23)} = \frac{31}{23} - \frac{1225/12167}{2354} 529 \frac{71749}{54142} \approx 1.3252.$$

We can notice a couple of things here. the first is that the numerators $f(x_i)$ are getting closer and closer to zero. This is what we should expect: we're trying to get closer and closer to a root of $f$.

Second, each successive step is smaller. From $x_0$ to $x_1$ we change by .5; from $x_1$ to $x_2$ we change by about 1.5; from $x_2$ to $x_3$ we change by about .02, which means we're probably within .02 of the true answer at $x_3$.

**Example 4.7.** Suppose we want to find a solution to $x^5 + x^2 + x - 1 = 0$. If we take $f(x) = x^5 + x^2 + x - 1$, then $f(0) = -1$ and $f(1) = 2$ so there must be at least one solution to this equation. But a result from the field of Galois theory tells us that we cannot express the solution exactly.

However, we can use Newton's method. $f(0) = -1$ so it seems reasonable to start with 0 as a guessed root. We compute $f'(x) = 5x^4 + 2x + 1$, and so if $x_0 = 0$ we have

$$x_1 = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{-1}{1} = 1$$
$$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{2}{8} = \frac{3}{4}$$
$$x_3 = \frac{3}{4} - \frac{f(3/4)}{f'(3/4)} \approx .75 - \frac{563/1024}{1045/256} = \frac{643}{1045} \approx .615311.$$

If we keep going, we see the true root is about $x = .586544$.