Math 1231: Single-Variable Calculus 1
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Recitation 1

Jay Daigle

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In class we talked about estimating functions and controlling their error margins. We had a target output, and an allowed error margin ε. Then we wanted to find the error δ we could permit in the input to keep our output from getting too far off.

Problem 1 (Warmup)**.** A cylindrical water tank has a base with an area of six square inches. We want to fill it with water ten inches deep, to the nearest inch.

1. What is the exact volume of water would we ideally want to pour in?
2. What ε do we want for our error margin?
3. What possible volumes of water will allow us to stay within our error margin?
4. What δ does that give us?

Problem 2. In class we talked about the square root function. We know that $\sqrt{4} = 2$. Suppose we take $\varepsilon = 1$ so we want our output to be 2 ± 1.

1. What is the largest input that keeps the output within ε of 2?
2. What is the smallest input that keeps the output within ε of 2?
3. What does that make δ?

Now instead let's take $\epsilon = .5$.

4. What is the largest input that keeps the output within ε of 2?

5. What is the smallest input that keeps the output within ε of 2?

6. What does that make δ?

In class, we looked at the following question: Suppose we want to make a square platform that's 16 square meters, plus or minus 1. How long do the sides need to be?

Clearly, our sides need to be between $\sqrt{15}$ and $\sqrt{17}$ but that doesn't tell us anything useful. So instead we made the following argument: We can use an absolute value to describe the way we think about errors. in particular, what we want here is

$$|s^2 - 16| < \varepsilon = 1, \tag{1}$$

and factoring the left hand side gives $|s-4| \cdot |s+4| < 1$. We can't solve this exactly, but we can make the following lazy decision: We know s should be *approximately* 4. It might be a little bigger, so $s+4$ might be bigger than 8, but it's certainly less than 9, or 10. Then we just need to solve

$$|s-4| \cdot |s+4| < 10|s-4| < 1 \tag{2}$$

$$|s-4| < .1 \tag{3}$$

$$-.1 < s - 4 < .1 \tag{4}$$

$$3.9 < s - 4 < 4.1. \tag{5}$$

Thus $\delta = .1$ and s should be $4 \pm .1$.

This is a tricky argument! But I want you to try to think through it now.

Problem 3. Let's suppose instead we want to make a square platform with area 25 square meters, plus or minus 1.

1. Write down the analogue of inequality (1) for this new problem. Can you explain in words what this inequality says about your error?

2. We can factor the left-hand side of this inequality into two factors. If our input is close to 5, one of these terms will be small, and the other will be large. Which one will be large, and about how large will that be?

3. This should let you write down an inequality like the one in (2). What is it?

4. Figure out δ such that $s = 5 \pm \delta$ will keep us in our error bounds.

5. Check your answer: square $5 + \delta$ and $5 - \delta$ and see whether the answers fall within your error margin.

6. Could you use a larger δ than the one you found in part (4)?

Problem 4. If time permits: redo 3 with $\varepsilon = .1$. You'll notice that you can do this pretty quickly, since you already did the hard part. If we change ε again, it should be easy to find a new δ.

Problem 5. Let $f(x) = 5x + 2$. We want to use an $\varepsilon - \delta$ argument to compute $\lim_{x \to 2} f(x)$.

1. If x is about 2, what should $f(x)$ be?

2. Write down expressions using absolute value for the input and output errors.

3. If we want $\varepsilon = 1$, what does δ need to be?

4. Find a formula for δ in terms of ε (same form as $\delta = \varepsilon/3$ or $\delta = \varepsilon$).

5. Try to write a full proof.

Problem 6. Let $g(x) = x^2$. We want to use an $\varepsilon - \delta$ argument to compute $\lim_{x \to 0} g(x)$.

1. If x is about 0, what should $g(x)$ be?

2. Write down expressions using absolute value for the input and output errors.

3. If we want $\varepsilon = 1$, what does δ need to be? What about $\varepsilon = 1/4$?

4. Find a formula for δ in terms of ε (same form as $\delta = \varepsilon/3$ or $\delta = \varepsilon$).

5. Try to write a full proof.