Math 1231 Spring 2024
Single-Variable Calculus I Section 11
Mastery Quiz 10
Due Tuesday, April 2

This week's mastery quiz has three topics. If you have a 4/4 on M3, or a 2/2 on S7 or S8, you don't need to submit them. (Yes, these are all the same topics as last week.)

Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Thursday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Optimization and Extrema
- Secondary Topic 7: Curve Sketching
- Secondary Topic 8: Physical Optimization

Name:

Recitation Section:

Major Topic 3: Optimization and Extrema

(a) The function $g(x) = x^3 - 3x^2 - 9x + 3$ has absolute extrema either on the interval $(-2, 4)$ or on the interval $[-2, 4]$. Pick one of those intervals, explain why g has extrema on that interval, and find the absolute extrema.

Solution: g is continuous on the closed interval $[-2, 4]$ so by the extreme value theorem it has an absolute maximum and an absolute minimum.

We compute $g'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$ is always defined, and is zero if $x = -1$ or $x = 3$. So the critical points are -1 and 3, and we need to check the points $-2, -1, 3, 4$.

$$g(-2) = 1 \qquad g(-1) = 8$$
$$g(3) = -24 \qquad g(4) = -17.$$

Thus g has a maximum of 8 at -1 and a minimum of -24 at 3.

(b) Classify the critical points and relative extrema of $g(x) = \cos^2(x) - 2\sin(x)$ on $[0, 2\pi]$.

Solution: We have

$$g'(x) = -2\cos(x)\sin(x) - 2\cos(x) = -2\cos(x)(\sin(x) + 1)$$

so $g'(x)$ is defined everywhere, and is 0 at $\pi/2, 3\pi/2$.

Here it looks easiest to use the second derivative test. We compute:

$$\begin{aligned} g''(x) &= 2\sin^2(x) - 2\cos^2(x) + 2\sin(x) \\ g''(\pi/2) &= 2 - 0 + 2 = 4 > 0 \\ g''(3pi/2) &= 2 - 0 - 2 = 0. \end{aligned}$$

So this tells us that g has a local minimum at $\pi/2$, but doesn't tell us what happens at $3\pi/2$.

To answer that question we need the first derivative. If we make a chart we can plug in values like

$$\begin{aligned} g'(0) &= -2 \\ g'(\pi) &= 2 \\ g''(2\pi) &= -2 \end{aligned}$$

	$g'(x)$
$0 \leq x < \pi/2$	$-$
$\pi/2 < x < 3\pi/2$	$+$
$3\pi/2 < x \leq 2\pi$	$-$

so g has a relative minimum at $\pi/2$ and a relative maximum at $3\pi/2$.

Secondary Topic 7: Curve Sketching

Sketch the graph of $f(x) = x^5 - 5x^4 + 5x^3 = x^3(x^2 - 5x + 5)$. We have $f'(x) = 5x^2(x-3)(x-1)$ and $f''(x) = 10x(2x^2 - 6x + 3)$.

You should discuss the domain, limits at infinity, critical points, intervals of increase and decrease, concavity, and possible points of inflection.

Solution: The domain of f is all reals.

There are roots at 0 and at $5/2 \pm \sqrt{5}/2$.

We see that $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

The critical points are $0, 1, 3$. We compute $f(0) = 0, f(1) = 1, f(3) = -27$.

For increase and decrease we make a chart:

	$5x^2$	$(x-3)$	$(x-1)$	$f'(x)$
$x < 0$	$+$	$-$	$-$	$+$
$0 < x < 1$	$+$	$-$	$-$	$+$
$1 < x < 3$	$+$	$-$	$+$	$-$
$3 < x$	$+$	$+$	$+$	$+$

Thus f is increasing on $(-\infty, 1)$ and on $(3, +\infty)$, and is decreasing on $(1, 3)$.

The possible points of inflection are 0 and $\frac{6\pm\sqrt{36-24}}{4} = \frac{3\pm\sqrt{3}}{2}$. We can make a chart:

	$10x$	$2x^2 - 6x + 3$	$f'(x)$
$x < 0$	$-$	$+$	$-$
$0 < x < (3-\sqrt{3})/2$	$+$	$+$	$+$
$(3-\sqrt{3})/2 < x < (3+\sqrt{3}))/2$	$+$	$-$	$-$
$(3+\sqrt{3})/2 < x$	$+$	$+$	$+$

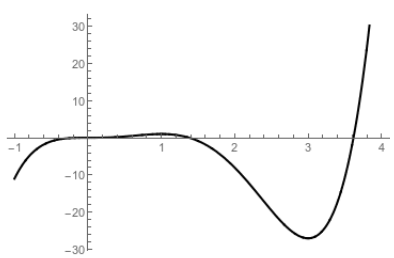


Figure 1: Graph of $f(x)$

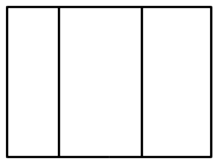
Secondary Topic 8: Physical Optimization

We wish to build a rectangular pen with two parallel internal partitions, using 1000 feet of fencing. We want to maximize the total area of the pen.

(a) What is your objective function, and why?

(b) What constraint equation(s) can you use?

(c) What dimensions maximize the total area of the pen?



Solution: Our objective function is $A = \ell w$, because this is the area we want to maximize. We see also that we have the constraint $2\ell + 4w = 1000$ so we can write $\ell = 500 - 2w$, and thus we have

$$A = (500 - 2w)w = 500w - 2w^2$$
$$A' = 500 - 4w$$

has a critical point when $w = 125$.

We can see this is a maximum using the extreme value theorem: the function is defined on the interval $[0, 250]$, and $A(0) = A(250) = 0$.

Or we can use the first derivative test; we see that $A'(w) < 0$ when $w > 125$ and $A'(w) > 0$ when $w < 125$, so A has a local maximum at $w = 125$.

Or we can use the second derivative test. $A'' = -4 < 0$ so we have a local maximum.

Thus the pen is maximized with width 125 and length 250. (The maximum area, which I didn't ask for, is 31250 square feet.)