Math 1231-13: Single-Variable Calculus 1
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Recitation 10

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Problem 1. If we have 1200 cm^2 of cardboard to make a box with a square base and an open top, what is the largest possible volume of the box?

(a) What are we trying to do? What is our objective function?

(b) What constraints are we acting under?

(c) Use our constraints to get a single-variable function to optimize.

(d) Find the optimum value.

(e) How do we know this is really the largest possible asnwer?

Solution:

(a) We want to maximize the volume of the box, which is given by $V = \ell w h$.

(b) First, we know the box has a square base, so $\ell = w$. We know that the total surface area of the box is $A = 1200$, and we also know that if the height of the box is h and the length of the base sides is wb, then the area is $A = w^2 + 4wh$. So we have

$$1200 = w^2 + 4wh.$$

(c) We can write $h = \frac{1200-w^2}{4w}$, and thus we have

$$V = wb^2h = w^2\frac{1200 - w^2}{4w}.$$

(d)

$$\begin{aligned}
V &= wb^2h = w^2\frac{1200 - w^2}{4w} \\
&= 300w - w^3/4 \\
V' &= 300 - 3w^2/4 \\
300 &= 3w^2/4 \\
400 &= w^2 \\
20 &= w
\end{aligned}$$

so the only critical point occurs at $w = 20$. We see that $V(20) = 400 \cdot 10 = 4000$, so this is the largest possible volume of the box.

(e) We can't actually use the extreme value theorem here: the width can be very small while satisfying the constraints, but it can't actually be zero.

So we have two options. One is to notice that $V' = 300 - 3w^2/4$ will be positive if w is small and positive, and negative if w is big. (We could make a chart; or we could see that $V'(2) = 297 > 0$, and $V'(40) = -900 < 0$.) Since V is always increasing from zero to 20, and then decreasing afterwards, we know we must have a maximum at 0.

Alternatively, we could see that $V''(w) = -3w/2$. When w is positive, this is always negative. That means the function is concave down everywhere, and so this should be a maximum.

Problem 2. A pizzeria sells pizzas for \$10 per pizza, and it costs $2x + x^2$ cents to make x pizzas. How many pizzas should the pizzeria make to maximize profit, and how much profit will it make?

(a) What is your objective function?

(b) Is there a constraint here? What?

(c) Make a single-variable function and find the critical points.

(d) Answer the question. How do we know when we have a maximum or minimum?

Solution:

(a) The objective function is profit, which is revenue minus costs. $P = R - C$.

(b) The constraint is, basically, that we can't sell more pizzas than we make. So our revenue is $10x$ and our costs are $\frac{2x+x^2}{100}$.

(c) Thus our profit is $P(x) = 10x - \frac{1}{100}(2x + x^2) = 9.98x - .01x^2$. Then

$$P'(x) = 9.98 - .02x$$
$$2x = 998$$
$$x = 499$$

and the critical point is $x = 499$.

(d) This suggests we should make 499 pizzas, for a total profit of $4990 - 9.98 - 499^2/100 = 2490.01$.

We can tell this profit is a maximum by looking at the first derivative: if we make fewer than 499 pizzas the derivative is positive, and if we make more the derivative is negative. So our profit is continually increasing as we make more pizzas until we reach 499, at which point it starts decreasing.

We could also consider the second derivative $P''(x) = -.02$ which is always negative. This tells us the function is always concave down, and thus should have a single maximum.

Problem 3. A piece of wire 10 m long is going to be cut into two pieces. We will fold one piece into a square and the other into an equilateral triangle. What is the largest joint area we can enclose? What is the smallest?

(a) What is your objective function? Do you need one objective function, or two?

(b) What constraint are you operating under?

(c) Make a single-variable function and find the critical points.

(d) Answer the questions. How do we know when we have a maximum or minimum?

Solution:

(a) The area of the square is $A_1 = s^2$, and the area of the triangle is $A_2 = bh/2$. So the objective function is $A = s^2 + bh/2$. Note this function is, like, completely useless because it has three variables in it.

We do only need one objective function, though—we want both the maximum and the minimum of the same function.

(b) If L is the length of wire into a triangle, and S is the length bent into a square, then $L + S = 10$ so $S = 10 - L$. Then the side length of a square is $\frac{10-L}{4}$.

The side length of the triangle is $L/3$, and the height is

$$\sin(\pi/3) \cdot L/3 = (1/2) \cdot (\sqrt{3}/2) \cdot L/3 = \sqrt{3}L/12.$$

(c) The area of the square is $(10 - L)^2/16$ and the area of the triangle is $L^2\sqrt{3}/36$. Then we have

$$\begin{aligned}
A = A_1 + A_2 &= (100 - 20L + L^2)/16 + L^2\sqrt{3}/36 \\
A' &= -5/4 + L/8 + L\sqrt{3}/18 \\
5/4 &= L/8 + L\sqrt{3}/18 \\
90 &= 9L + 4\sqrt{3}L \\
L &= 90/(9 + 4\sqrt{3})
\end{aligned}$$

This is the only critical point. At that point,

$$A \approx 1.2 + 1.5 = 2.7.$$

(d) Just finding the critical point and value isn't enough. We need a max and a min, and the single critical point is at most one of those. It's also not obvious *just* from the work we've done whether this is a maximum or a minimum. (Take a moment to think about which it should be, though!)

But here we can use the Extreme Value Theorem, since we could make just a square, or just a triangle. So this function is defined on $[0, 10]$, and it is continuous. If we use all the wire for the square so that $L = 0$, we have area $A = 100/16 = 6.25$. And if we use all the wire for the triangle so that $L = 10$, we have $A = 100\sqrt{3}/36 \approx 4.8$.

So we get the biggest area when we use all the wire for the square, and the smallest if we use $90/(9 + 4\sqrt{3})$m of wire for the triangle.