Math 1231 Spring 2024
Single-Variable Calculus I Section 11
Mastery Quiz 11
Due Tuesday, April 9

This week's mastery quiz has two topics. Everyone should submit S9. If you have a 4/4 on M3, you don't need to submit it again.

Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Thursday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Optimization and Extrema
- Secondary Topic 9: Riemann Sums

Name:

Recitation Section:

Major Topic 3: Optimization and Extrema

(a) The function $f(x) = \dfrac{x}{x^2+1}$ has absolute extrema either on the interval $[0,3]$ or on the interval $(2,4)$. Pick one of those intervals, explain why f has extrema on that interval, and find the absolute extrema.

Solution: f is continuous on a closed interval, so it must have an absolute max and min by the EVT.

$$f'(x) = \frac{1(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2}.$$

The denominator is never zero; the numerator is zero for $x = \pm 1$ so the critical points are ± 1. But -1 is irrelevant. We compute

$$f(0) = 0$$
$$f(1) = \frac{1}{2}$$
$$f(3) = \frac{3}{10}.$$

Thus the absolute maximum is 1/2, at 1; and the absolute minimum is 0, at 0.

(b) Classify the critical points and relative extrema of $f(x) = 5 + 8x^3 + x^4$.

Solution: We have $f'(x) = 24x^2 + 4x^3 = 4x^2(6+x)$. So the critical points are $-6, 0$. We could try the second derivative test: we get $f''(x) = 48x + 12x^2$. Then $f''(-6) = 12(-24+36) = 144 > 0$, so this is a relative minimum. But $f''(0) = 0$ which doesn't tell us anything. We have to pass to the first derivative test.

We make a chart:

	$4x^2$	$6+x$	$f'(x)$
$x < -6$	$+$	$-$	$-$
$-6 < x < 0$	$+$	$+$	$+$
$0 < x$	$+$	$+$	$+$

Thus f is decreasing when $x < -6$ and increasing when $x > -6$, and so we see that f has a minimum at $x = -6$, and a point which is neither a maximum nor a minimum at $x = 0$,.

Secondary Topic 9: Riemann Sums

Let $f(x) = 2x + 4x^2$ be defined on the interval $[0,2]$.

(a) Approximate the area under the curve of the function using four rectangles and right endpoints.

(b) Approximate the area under the curve of the function using four rectangles and left endpoints.

(c) Write a formula for R_n, the estimate using n rectangles and right endpoints, as a summation of n terms.

(d) Use your answer in part (c) to find a closed-form formula for R_n. (This formula should not have a summation sign or be given as a sum of n terms.)

(e) Use the formula in part (c) to compute the area exactly.

Solution:

(a) $R_4 = 1/2 \cdot f(1/2) + 1/2 \cdot f(1) + 1/2 \cdot f(3/2) + 1/2 \cdot f(2) = \frac{1}{2}(2 + 6 + 12 + 20) = 20$

(b) $L_4 = 1/2 \cdot f(0) + 1/2 \cdot f(1/2) + 1/2 \cdot f(1) + 1/2 \cdot f(3/2) = \frac{1}{2}(0 + 2 + 6 + 12) = 10$

(c)

$$R_n = \sum_{i=1}^{n} \frac{2}{n} f\left(0 + i\frac{2}{n}\right) = \sum_{i=1}^{n} \frac{2}{n}\left(2(2i/n) + 4(2i/n)^2\right)$$

(d)

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} \frac{2}{n}\left(2(2i/n) + 4(2i/n)^2\right) \\
&= \sum_{i=1}^{n} \frac{2}{n}\left(\frac{4i}{n} + \frac{16i^2}{n^2}\right) \\
&= \sum_{i=1}^{n} \frac{8i}{n^2} + \frac{32i^2}{n^3} \\
&= \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{32}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{8}{n^2}\frac{n(n+1)}{2} + \frac{32}{n^3}\frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

(e) We can compute

$$\begin{aligned}
\lim_{n\to+\infty} R_n &= \lim_{n\to+\infty} \frac{8}{n^2}\frac{n(n+1)}{2} + \frac{32}{n^3}\frac{n(n+1)(2n+1)}{6} \\
&= 4 + \frac{32}{3} = \frac{44}{3}.
\end{aligned}$$