Math 1231-13: Single-Variable Calculus 1
George Washington University   Spring 2024
Recitation 11

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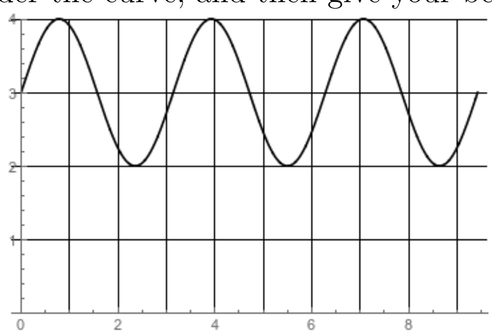
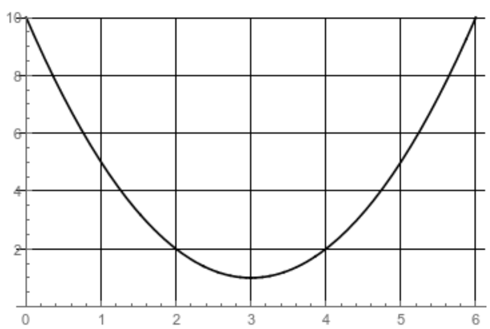
Problem 1. For the following curves, find an upper bound and a lower bound for the area under the curve, and then give your best estimate for the actual area.



(between 0 and 9; ignore the trailing bit off the right edge)

Solution: Upper bound: 36;
Lower bound: 18;
exact value: $9\pi \approx 28$



(between 0 and 6; ignore the trailing bit off the right edge)

Solution: Upper bound: 36;
Lower bound: 12;
exact value: 24

Problem 2. Consider the function $f(x) = \sqrt{1 - x^2}$ between $x = 0$ and $x = 1$.

(a) Estimate the area using two rectangles with right endpoints. Is this an upper bound, a lower bound, or neither?

(b) Estimate the area using two rectangles with left endpoints. Is this an upper bound, lower bound, or neither?

(c) Find an upper bound using four rectangles.

(d) Find a lower bound using four rectangles.

(e) Can you guess what the area under the curve is exactly? (Hint: what does the graph look like?)

Solution:

(a)
$$R_2 = \frac{1}{2}\left(\sqrt{1-1/4}+0\right) = \frac{\sqrt{3}}{4} \approx .43.$$
This is a lower bound, since the function is decreasing and the right endpoint is always the lowest point in the interval.

(b)
$$L_2 = \frac{1}{2}\left(1+\sqrt{1-1/4}\right) = \frac{4+\sqrt{3}}{4} \approx .93.$$
This is an upper bound, since the function is decreasing and the left endpoint is always the highest point in the interval.

(c)
$$L_4 = \frac{1}{4}\left(1+\sqrt{1-1/16}+\sqrt{1-1/4}+\sqrt{1-9/16}\right) = \frac{4+\sqrt{15}+2\sqrt{3}+\sqrt{7}}{16} \approx .87.$$

(d)
$$R_4 = \frac{1}{4}\left(\sqrt{1-1/16}+\sqrt{1-1/4}+\sqrt{1-9/16}+0\right) = \frac{\sqrt{15}+2\sqrt{3}+\sqrt{7}}{16} \approx .62.$$

(e) This is the graph of a quarter circle—in fact, the upper-right quadrant of the unit circle—so the area should be $\pi/4 \approx .79$.

Problem 3. Consider the function $g(x) = x^3$ between $x = 0$ and $x = 1$.

(a) Estimate the area using two rectangles with right endpoints. Is this an upper bound, a lower bound, or neither?

(b) Estimate the area using two rectangles with left endpoints. Is this an upper bound, lower bound, or neither?

(c) Find an upper bound using four rectangles.

(d) Find a lower bound using four rectangles.

(e) Find a formula using right endpoints to estimate the area using n rectangles, in summation form.

(f) Use your summation rules to get a closed-form formula, with no summation signs in it.

(g) Take a limit to find the exact area under this curve. (Use your summation rules!)

Solution:

(a)
$$R_2 = \frac{1}{2}\left(\frac{1}{2}^3 + 1^3\right) = \frac{9}{16}$$
This is an upper bound, since the function is increasing.

(b)
$$L_2 = \frac{1}{2}\left(0^3 + \frac{1}{2}^3\right) = \frac{1}{16}.$$
This is a lower bound, since the function is decreasing.

(c)
$$R_4 = \frac{1}{4}\left(\frac{1}{4}^3 + \frac{1}{2}^3 + \frac{3}{4}^3 + 1^3\right) \approx .39.$$

(d)
$$L_4 = \frac{1}{4}\left(0 + \frac{1}{4}^3 + \frac{1}{2}^3 + \frac{3}{4}^3 + \right) \approx .14$$

(e)
$$R_n = \sum_{i=1}^n f\left(0 + i\frac{1-0}{n}\right) \cdot \frac{1-0}{n} = \sum_{i=1}^n \left(\frac{i}{n}\right)^3 \cdot \frac{1}{n}.$$

$$\begin{aligned} R_n &= \frac{1}{n}\sum_{i=1}^n \frac{i}{n}^3 = \frac{1}{n}\frac{1}{n^3}\sum_{i=1}^n i^3 \\ &= \frac{1}{n^4}\left(\frac{(n)(n+1)}{2}\right)^2 = \frac{n^4+2n^3+n^2}{4n^4}. \end{aligned}$$

(f)
$$\lim_{n\to+\infty} R_n = \lim_{n\to+\infty} \frac{n^2+2n+1}{4n^2} = \lim_{n\to+\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4} = \frac{1}{4}.$$

Problem 4. Suppose we know that $\int_2^4 f(x)\,dx = 3$, $\int_4^6 f(x)\,dx = 5$, and $\int_2^6 g(x)\,dx = -2$. Compute the following integrals, justifying your answers:

(a) $\int_2^4 3f(x)\,dx$?

(b) $\int_2^6 f(x) - g(x)\,dx$?

(c) $\int_6^4 f(x) - 3\,dx$?

Solution:

(a) $\int_2^4 3f(x)\,dx = 3\int_2^4 f(x)\,dx = 9$.

(b) $\int_2^6 f(x) - g(x)\,dx = \int_2^4 f(x)\,dx + \int_4^6 f(x)\,dx - \int_2^6 g(x)\,dx = 3 + 5 + 2 = 10$.

(c) $\int_6^4 f(x) - 3\,dx = \int_4^6 3\,dx - \int_4^6 f(x)\,dx = 6 - 10 = -4$.

Problem 5. (a) Use the Fundamental Theorem of Calculus to compute $\frac{d}{dx}\int_2^x \sqrt{t^5 - t}\,dt$.

(b) Compute $\frac{d}{dx}\int_x^5 s^5 + \cos(s^2)\,ds$. What rule did you have to use here other than the FTC?

(c) Compute $\frac{d}{dx}\int_{-3}^{x^2} \sqrt{t^3 + 1}\,dt$. What rule did you have to use here other than the FTC?

Solution:

(a)
$$\frac{d}{dx}\int_2^x \sqrt{t^5 - t}\,dt = \sqrt{x^5 - x}.$$

(b)
$$\frac{d}{dx}\int_x^5 s^5 + \cos(s^2)\,ds = \frac{d}{dx} - \int_5^x s^5 + \cos(s^2)\,ds = -x^5 - \cos(x^2).$$

We had to use the integrals rule that allows us to flip the bounds!

(c) This one is tricky becayse the upper bound isn't just an x. We need to use the chain rule here.

We know that
$$\frac{d}{dx}\int_{-3}^x \sqrt{t^3 + 1}\,dt = \sqrt{x^3 + 1}.$$

We can write $G(x) = \int_{-3}^x \sqrt{t^3 + 1}\,dt$ and then $G'(x) = \sqrt{x^3 + 1}$; we're looking to find $\frac{d}{dx}G(x^2)$. But by the chain rule this is just $G'(x^2) \cdot 2x$, so we compute

$$\frac{d}{dx}\int_{-3}^{x^2} \sqrt{t^3}\,dt = \sqrt{x^6 + 1} \cdot 2x.$$

Problem 6. We want to find $\frac{d}{dx}\int_{3x}^{x^3} \sqrt[3]{x+1}\,dx$. Unfortunately we can't apply the Fundamental Theorem of Calculus directly.

(a) This integral has variables in both the upper and lower bounds. Can you split it into multiple integrals, each of which has only one variable in a bound?

(b) To use the FTC we need the variable as the *upper* bound of each integral. How can we do that?

(c) Now for each integral you have set up, carefully take the derivative, paying attention to the chain rule.

(d) Combine this work to answer the original question.

Solution:

(a) We have
$$\int_{3x}^{x^3} \sqrt[3]{x+1}\,dx = \int_{3x}^{0} \sqrt[3]{x+1}\,dx + \int_{0}^{x^3} \sqrt[3]{x+1}\,dx.$$
Note that the choice of constant doesn't matter; you can pick anything there.

(b)
$$\int_{3x}^{0} \sqrt[3]{x+1}\,dx = -\int_{0}^{3x} \sqrt[3]{x+1}\,dx.$$

(c)
$$\frac{d}{dx}\int_{0}^{3x} \sqrt[3]{x+1}\,dx = \sqrt[3]{3x+1}\cdot 3$$
$$\frac{d}{dx}\int_{0}^{x^3} \sqrt[3]{x+1}\,dx = \sqrt[3]{x^3+1}\cdot 3x^2$$

(d)
$$\frac{d}{dx}\int_{3x}^{x^3} \sqrt[3]{x+1}\,dx = \sqrt[3]{x^3+1}\cdot 3x^2 - \sqrt[3]{3x+1}\cdot 3.$$