Math 1231-13: Single-Variable Calculus 1
George Washington University Spring 2024
Recitation 11

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Problem 1. Compute the following integrals, without using substitutions:

(a) $\displaystyle\int x(x+1)\,dx$

Solution: $\int x(x+1)\,dx = \int x^2 + x\,dx = \frac{x^3}{3} + \frac{x^2}{2} + C$.

We always like polynomials when we can have them.

(b) $\displaystyle\int x\sqrt{x}\,dx$

Solution: $\int x\sqrt{x}\,dx = \int x^{3/2} = \frac{x^{5/2}}{5/2} + C = \frac{2x^2\sqrt{x}}{5} + C$.

Writing things as pure fractional exponents almost always makes them easy to deal with.

(c) $\displaystyle\int 5\csc(x)\cot(x)\,dx$

Solution:

$$\int 5\csc(x)\cot(x)\,dx = -5\csc(x) + C.$$

This expression looks complicated but it's secretly not.

(d) $\displaystyle\int (x^4 - x)(x^2 + x + 1)\,dx.$

Solution:

$$\begin{aligned}\int (x^4 - x)(x^2 + x + 1)\,dx &= \int x^8 + x^5 + x^4 - x^3 - x^2 - x\,dx \\ &= \frac{1}{9}x^9 + \frac{1}{6}x^6 + \frac{1}{5}x^5 - \frac{1}{4}x^4 - \frac{1}{3}x^3 - \frac{1}{2}x^2 + C.\end{aligned}$$

Problem 2. Compute the following integrals:

(a) $\int \sqrt{3x-4}\,dx$.

Solution: We can take $u = 3x - 4$, which is the inside function. Then $du = 3\,dx$ so we have

$$\begin{aligned}\int \sqrt{3x-4}\,dx &= \int \sqrt{u}\frac{du}{3} = \frac{1}{3}\int \sqrt{u}\,du \\ &= \frac{1}{3}\int u^{1/2}\,du = \frac{1}{3}\frac{u^{3/2}}{3/2} + C \\ &= \frac{1}{3}\frac{(3x-4)^{3/2}}{3/2} + C.\end{aligned}$$

(b) $\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$.

Solution: Set $u = \sqrt{x}$, and $du = \frac{dx}{2\sqrt{x}}$. Thus

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}} = \int 2\sin(u)\,du = -2\cos(u) + C = -2\cos(\sqrt{x}) + C.$$

(c) $\int x\sqrt{x+1}\,dx$.

Solution: Take $u = x + 1$, $du = 1 \cdot dx$. Then

$$\begin{aligned}\int x\sqrt{x+1}\,dx &= \int (u-1)\sqrt{u}\,du = \int u^{3/2} - u^{1/2}\,du \\ &= \frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3} + C \\ &= \frac{2(x+1)^{5/2}}{5} - \frac{2(x+1)^{3/2}}{3} + C.\end{aligned}$$

Problem 3. (a) Compute $\int_1^2 \frac{6x^2-7}{\sqrt{2x^3-7x+14}}\,dx$ using a u-substitution and explicitly changing the bounds of integration.

(b) Now compute the indefinite integral $\int \frac{6x^2-7}{\sqrt{2x^3-7x+14}}\,dx$.

(c) Use your answer in part (b) to compute $\int_1^2 \frac{6x^2-7}{\sqrt{2x^3-7x+14}}\,dx$ again. How does this compare to what you did in part (a)?

Solution:

(a) We take $u = 2x^3 - 7x + 14$, so $du = (6x^2 - 7)\,dx$. Then $u(1) = 9$ and $u(2) = 16$. So we have

$$\begin{aligned}\int_1^2 \frac{6x^2-7}{\sqrt{2x^3-7x+14}}\,dx &= \int_9^{16} \frac{6x^2-7}{\sqrt{u}}\frac{du}{6x^2-7} \\ &= \int_9^{16} u^{-1/2}\,du = 2u^{1/2}\Big|_9^{16} = 8 - 6 = 2.\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{6x^2-7}{\sqrt{2x^3-7x+14}}\,dx &= \int \frac{6x^2-7}{\sqrt{u}}\frac{du}{6x^2-7} \\ &= \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{2x^3-7x+14} + C.\end{aligned}$$

(c)

$$\begin{aligned}\int_1^2 \frac{6x^2-7}{\sqrt{2x^3-7x+14}}\,dx &= 2\sqrt{2x^3-7x+14}\Big|_1^2 \\ &= 2\sqrt{16} - 2\sqrt{9} = 8 - 6 = 2.\end{aligned}$$

Notice that we wind up not only with the same answer, but the same final calculation, as in part (a); we're plugging the same values in to the same function $2x^3 - 7x + 14$, just in a more awkward spot.

Problem 4. We want to compute $\int \sec^8(x)\tan(x)\,dx$. Can you find multiple u that all work?

Solution: The "obvious" u to take is $u = \sec(x)$ so $du = \sec(x)\tan(x)dx$. Then

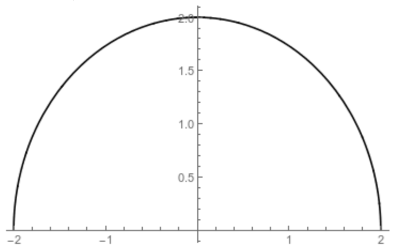
$$\int \sec^8(x)\tan(x)\,dx = \int u^7\,du = \frac{u^8}{8} + C = \frac{\sec^8(x)}{8} + C.$$

It also works to take $u = \sec^2(x)$ or $\sec^4(x)$ or $\sec^8(x)$. e.g. if $u = \sec^8(x)$, then $du = 8\sec^8(x)\tan(x)dx$ and

$$\int \sec^8(x)\tan(x)\,dx = \int \frac{1}{8}\,du = \frac{u}{8} + C = \frac{\sec^8(x)}{8} + C.$$

Problem 5. Evaluate $\int_{-2}^{2} 4\sqrt{4-x^2}\,dx$ by thinking about area. (Hint: what does the graph of $\sqrt{4-x^2}$ look like?)

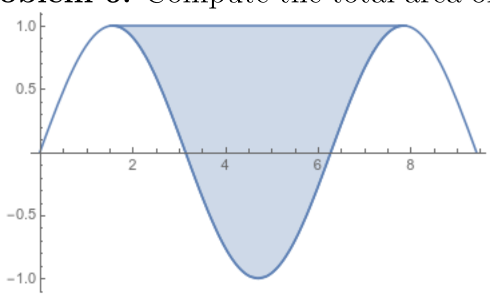
Solution: It's possible to do the integral algebraically, but not really with the techniques we have in this course. (The big idea from Calc 2 is we can do the substitution $x = \sin(u)$ and then use trigonometric identities to simplify the integral.) Instead, we graph the function $\sqrt{4-x^2}$, and see that it is a semicircle with radius 2, and thus has area $\pi(2^2)/2 = 2\pi$.



We multiply by four to get 8π, and thus have

$$\int_{-2}^{2} 4\sqrt{4-x^2}\,dx = 8\pi.$$

Problem 6. Compute the total area of the "valley" between two peaks of the sine function.



Solution: We see that this area is the area of the region between $y = 1$ and $y = \sin x$ between $\pi/2$ and $5\pi/2$. (There are other ways to set this up, but this way works). So we compute

$$\int_{\pi/2}^{5\pi/2} 1 - \sin x\,dx = x + \cos(x)\Big|_{\pi/2}^{5\pi/2} = (5\pi/2 + 0) - (\pi/2 + 0) = 2\pi.$$