Math 1231 Spring 2024
Single-Variable Calculus I Section 11
Mastery Quiz 13
Due Tuesday, April 23

This week's mastery quiz has two topics. Everyone should submit both M4 and S10.

Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Thursday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 4: Integration
- Secondary Topic 10: Integral Application

Name:

Recitation Section:

Major Topic 4: Integration

(a) By explicitly changing the bounds of integration, compute $\int_1^2 x^5\sqrt{x^3+8}\,dx$.

Solution: We take $u = x^3 + 8$, so we have $du = 3x^2\,dx, dx = \frac{du}{3x^2}$, and we have $g(1) = 9$ and $g(2) = 16$. Then we compute

$$\begin{aligned}
\int_1^2 x^5\sqrt{x^3+8}\,dx &= \int_9^{16} x^5\sqrt{u}\frac{du}{3x^2} \\
&= \frac{1}{3}\int_9^{16} x^3\sqrt{u}\,du = \frac{1}{3}\int_9^{16}(u-8)\sqrt{u}\,du \\
&= \frac{1}{3}\int_9^{16} u^{3/2} - 8u^{1/2}\,du \\
&= \frac{1}{3}\left(\frac{u^{5/2}}{5/2} - \frac{8u^{3/2}}{3/2}\right)\Bigg|_9^{16} \\
&= \frac{1}{3}\left(\left(\frac{2\cdot 4^5}{5} - \frac{8\cdot 2\cdot 4^3}{3}\right) - \left(\frac{2\cdot 3^5}{5} - \frac{8\cdot 2\cdot 3^3}{3}\right)\right) \\
&= \frac{2048}{15} - \frac{1024}{9} - \frac{162}{5} + 48 = \frac{1726}{45} = 38.355\ldots.
\end{aligned}$$

(b) Compute $\int x\cos(3x^2-2)\,dx =$

Solution: Set $u = 3x^2 - 2$ so $du = 6xdx$ and $dx = \frac{du}{6x}$. Then

$$\begin{aligned}
\int x\cos(3x^2-2)\,dx &= \int x\cos(u)\frac{du}{6x} = \frac{1}{6}\int\cos(u)\,du \\
&= \frac{1}{6}\sin(u) + C = \frac{1}{6}\sin(3x^2-2) + C.
\end{aligned}$$

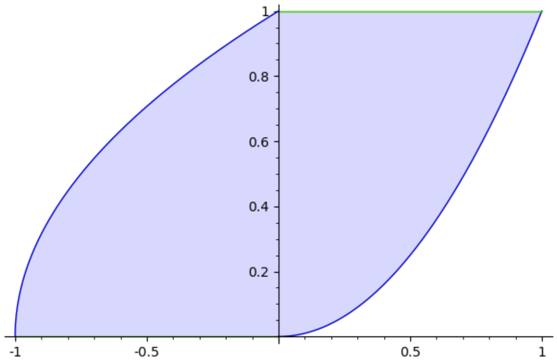
(c) Compute $\int \sin^4(t)\cos(t)\,dt$

Solution: We can take $u = \sin(t)$, then we have $du = \cos(t)\,dt$ so we are computing

$$\int u^4\,du = \frac{1}{5}u^5 + C = \frac{\sin^5(t)}{5} + C.$$

Secondary Topic 10: Integral Applications

(a) Sketch the region bounded by the curves $x = y^2 - 1$, $y = 0$, $y = 1$, and $x = \sqrt{y}$, and find the area of that region.



Solution:

We really want to integrate this with respect to y. So we have

$$\int_0^1 \sqrt{y} - (y^2 - 1)\, dy = \int 1 + \sqrt{y} - y^2\, dy$$
$$= y + \frac{2}{3}y^{3/2} - \frac{y^3}{3}\Big|_0^1 = 1 + \frac{2}{3} - \frac{1}{3} = \frac{4}{3}.$$

If we really want to, though, we can integrate with respect to x. We compute

$$A = \int_{-1}^0 \sqrt{x+1}\, dx + \int_0^1 1 - x^2\, dx$$
$$= \frac{2}{3}(x+1)^{3/2}\Big|_{-1}^0 + x - \frac{x^3}{3}\Big|_0^1$$
$$= \frac{2}{3} - 0 + 1 - \frac{1}{3} - 0 + 0 = \frac{4}{3}.$$

(b) What is the average value of the function $h(x) = x + \sqrt{x}$ on the interval $[1, 4]$?

Solution:

$$h_{ave} = \frac{1}{3}\int_1^4 x + \sqrt{x}\, dx$$
$$= \frac{1}{3}x^2/2 + \frac{2}{3}x^{3/2}|_1^4$$
$$= \frac{1}{3}(8 + 16/3 - 1/2 - 2/3) = \frac{8}{3} + \frac{16}{9} - \frac{1}{6} - \frac{2}{9} = \frac{73}{18}$$

(c) Suppose your velocity is given by $v(t) = 3x - x^2 + \sin(pix)$ miles per hour, where t is in hours. How far in total do you travel between noon (at time 0) and 3 pm? Give units.

Solution: Our velocity is given in miles per hour, as a function of hours. So we can compute the distance traveled by integrating

$$\begin{aligned}
D &= \int_0^3 3x - x^2 + \sin(\pi x)\,dx \\
&= 3\int_0^3 x\,dx - \int_0^3 x^2\,dx + \int_0^3 \sin(\pi x)\,dx \qquad \text{set } u = \pi x \\
&= 3\int_0^3 x\,dx - \int_0^3 x^2\,dx + \int_0^{3\pi} \sin(u)\frac{du}{\pi} \\
&= \frac{3}{2}x^2\Big|_0^3 - \frac{1}{3}x^3\Big|_0^3 + \frac{1}{\pi}(-\cos(u))\Big|_0^{3\pi} \\
&= \frac{27}{2} - \frac{27}{3} + \frac{1}{\pi}(1+1) \\
&= \frac{9}{2} + \frac{2}{\pi}.
\end{aligned}$$

This answer will be in miles, since we are integrating miles per hour with respect to hours.