Math 1231-13: Single-Variable Calculus 1
George Washington University   Spring 2024
Recitation 13

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Problem 1. We want to find the area of the region bounded by $y = x^2 + 1, y = 17 - x^2$, and $x = 1$, taking the side with $x \geq 1$.

(a) Sketch the region in question. Based on the picture, would you rather integrate with respect to x or to y? Discuss this with someone near you.

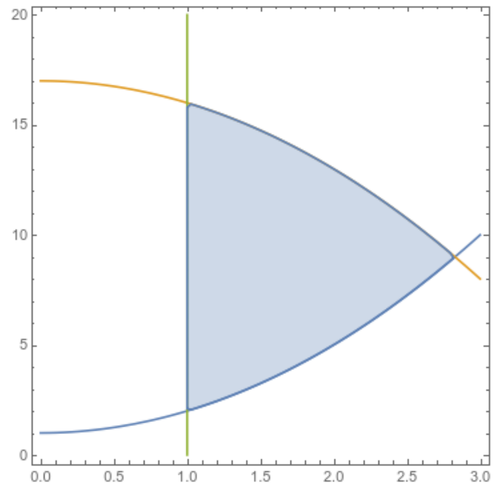
(b) Set up an integral to compute this region, integrating with respect to x.

(c) Set up an integral to compute this region, integrating with respect to y.

(d) Which of these integrals do you prefer? Pick one and compute it.

Solution:

(a) We first draw the region, and see a sort of sideways triangle with a base at $x = 1$ and a point where the curves $y = x^2 + 1$ and $y = 17 - x^2$ intersect. Setting them equal, we get $x^2 + 1 = 17 - x^2$, which gives $2x^2 = 16$ and $x = \pm\sqrt{8}$. Since $x \geq 1$ we know we want $x = \sqrt{8}$, and thus the point of the triangle is at $(\sqrt{8}, 9)$. Checking where the two curves hit $x = 1$ we see that y varies from 1 to 17.



(b) To integrate with respect to x, we see that x varies from 1 to $\sqrt{8}$, and get

$$A = \int_1^{\sqrt{8}} (17 - x^2) - (x^2 - 1)\, dx = \int_1^{\sqrt{8}} 18 - 2x^2\, dx.$$

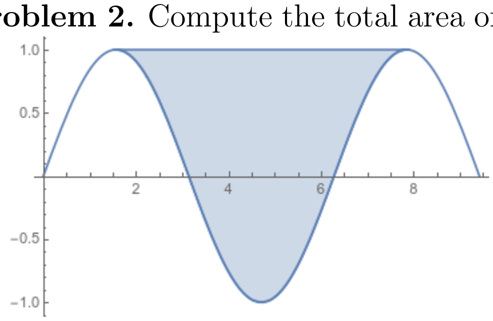
(c) To integrate with respect to y, we'd have to write x as a function of y: we see our two curves are $x = \sqrt{y-1}$ and $x = \sqrt{17-y}$. Then we have to break our region into two pieces: one as x goes from 2 to 9, and the other as x goes from 9 to 17.

$$A = \int_2^9 \sqrt{17-y} - 1\, dy + \int_9^{16} \sqrt{17-y} - 1\, dy,$$

(d) The second integral is in fact doable, but it's unnecessarily ugly. Instead we integrate with respect to x:

$$\begin{aligned} A &= \int_1^{\sqrt{8}} (17 - x^2) - (x^2 - 1)\, dx = \int_1^{\sqrt{8}} 18 - 2x^2\, dx \\ &= 18x - \frac{2}{3}x^3\Big|_1^{\sqrt{8}} = 36\sqrt{2} - 32\sqrt{2}/3 - 18 + 2/3 = \frac{76\sqrt{2} - 52}{3}. \end{aligned}$$

Problem 2. Compute the total area of the "valley" between two peaks of the sine function.



Solution: We see that this area is the area of the region between $y = 1$ and $y = \sin x$ between $\pi/2$ and $5\pi/2$. (There are other ways to set this up, but this way works). So we compute

$$\int_{\pi/2}^{5\pi/2} 1 - \sin x \, dx = x + \cos(x)\Big|_{\pi/2}^{5\pi/2} = (5\pi/2 + 0) - (\pi/2 + 0) = 2\pi.$$

Problem 3. For each of the following functions, figure out the units of $\int f(x)\,dx$. What is this integral computing:

(a) $f(x)$ gives acceleration in meters per second squared as a function of time in seconds.

(b) $f(x)$ gives tension in pounds per inch, as a function of how many inches along a material you are.

(c) $f(x)$ gives the pressure exerted by a gas (in newtons per square meter), as a function of the volume in cubic meters. (Imagine a piston moving out to expand a chamber full of gas under pressure.)

(d) $f(x)$ gives density in kilograms per meter, as a function of how many meters along a steel rod you are.

(e) $f(x)$ gives resistance in volts per ampere as a function of how many amperes you run through a wire.

Solution:

(a) $\int f(x)\,dx$ gives meters per second squared times seconds, which is meters per second. It gives your velocity as a function of time.

(b) $\int f(x)\,dx$ will give pounds per inch times inches, and thus pounds. It describes the total force applied to a material by that tension.

(c) $\int f(x)\,dx$ gives newtons per square meter, times cubic meters, which is newtons times meters. It's not obvious, but the work done by the gas as the volume changes!

(d) $\int f(x)\,dx$ gives you kilograms per meter times meters, which is just kilograms. It tells you the total mass of your object.

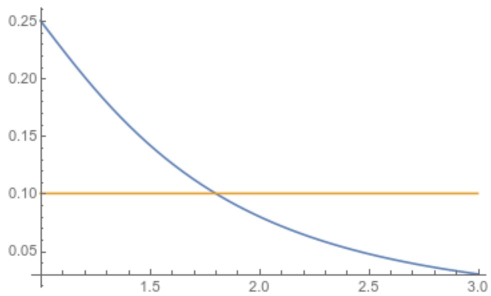
(e) $\int f(x)\,dx$ gives volts per amp times amps, or just volts. It computes the total voltage through your wire.

Problem 4. Find the average value of the function $\dfrac{x}{(x^2+1)^2}$ for $1 \leq x \leq 3$.

Solution: We take $u = x^2 + 1$ so $du = 2x\,dx$, and so

$$\begin{aligned} A &= \frac{1}{3-1}\int_1^3 \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\int_2^{10} \frac{x}{u^2}\cdot\frac{du}{2x} \\ &= \frac{1}{4}\int_2^{10} u^{-2}\,du = \frac{1}{4}(-u^{-1})\Big|_2^{10} \\ &= \frac{1}{4}\left(\frac{-1}{10} - \frac{-1}{2}\right) = \frac{1}{4}\cdot\frac{2}{5} = \frac{1}{10}. \end{aligned}$$

So the function has an average value of $\frac{1}{10}$ between 1 and 3.



Problem 5. Suppose the demand for pizzas is $D(q) = 25 - .0001q^2$ and the supply is $S(q) = 10 + .02q$.

(a) How many pizzas will be sold, and at what price?

(b) What is the consumer surplus?

(c) What is the producer surplus?

(d) What is the total surplus?

Solution:

(a) 300 pizzas, at a price of 16 dollars per pizza.

(b) $\int_0^{300} 25 - .0001q^2 - 16\,dq = 1800$.

(c) $\int_0^{300} 16 - (10 + .02q)\,dq = 900$.

(d) 2700.

Problem 6. A 12in spring is stretched to 15in by a force of 75lbs.

(a) What is the spring constant? What units does it have?

(b) What is the function that gives force as a function of position? what units does it have?

(c) What is the work done by stretching the spring from 16in to 20in? What units are your answer in?

Solution:

(a) We know that the force must be kx where x is the displacement. Since the displacement is 3in and the force is 75lbs we must have $k = 25$lbs/in.

(b) We have $F(12) = 0$ and $F(15) = 75$. We can write $F(x) = 25(x - 12)$. This function takes in inches, and outputs force in pounds.

(c) We need to integrate the force over the displacement we're using. So we compute

$$\begin{aligned} W &= \int_{16}^{20} 25(x-12)\,dx = \int_{16}^{20} 25x - 300\,dx \\ &= \frac{25}{2}x^2 - 300x\Big|_{16}^{20} = (5000 - 6000) - (3200 - 4800) \\ &= -1000 - (-1600) = 600. \end{aligned}$$

Thus the work is in lbs. The units are, importantly, foot-inches; in the more usual units of foot-pounds, the work is 50ft lbs.