Math 1231-13: Single-Variable Calculus 1
George Washington University Spring 2024
Recitation 14

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Problem 1. A 12in spring is stretched to 15in by a force of 75lbs.

(a) What is the spring constant? What units does it have?

(b) What is the function that gives force as a function of position? what units does it have?

(c) What is the work done by stretching the spring from 16in to 20in? What units are your answer in?

Solution:

(a) We know that the force must be kx where x is the displacement. Since the displacement is 3in and the force is 75lbs we must have $k = 25$lbs/in.

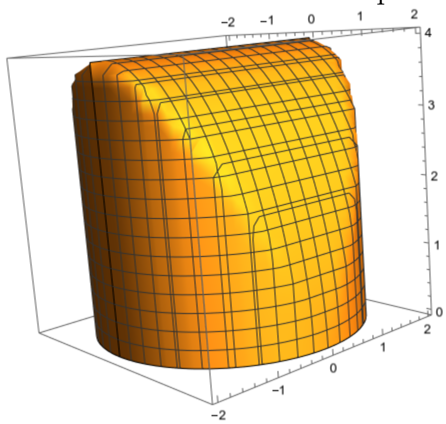
(b) We have $F(12) = 0$ and $F(15) = 75$. We can write $F(x) = 25(x - 12)$. This function takes in inches, and outputs force in pounds.

(c) We need to integrate the force over the displacement we're using. So we compute

$$\begin{aligned} W &= \int_{16}^{20} 25(x-12)\,dx = \int_{16}^{20} 25x - 300\,dx \\ &= \frac{25}{2}x^2 - 300x\Big|_{16}^{20} = (5000 - 6000) - (3200 - 4800) \\ &= -1000 - (-1600) = 600. \end{aligned}$$

Thus the work is in lbs. The units are, importantly, foot-inches; in the more usual units of foot-pounds, the work is 50ft lbs.

Problem 2. Find the volume of a shape whose base is a circle of radius 2, where slices perpendicular to the base are squares.



Problem 3. Let's find the volume of the solid generated by rotating the region bounded by $y = x$ and $y = \sqrt{x}$ about the line $y = 1$.

(a) Sketch a picture of the region, and draw in the line of revolution.

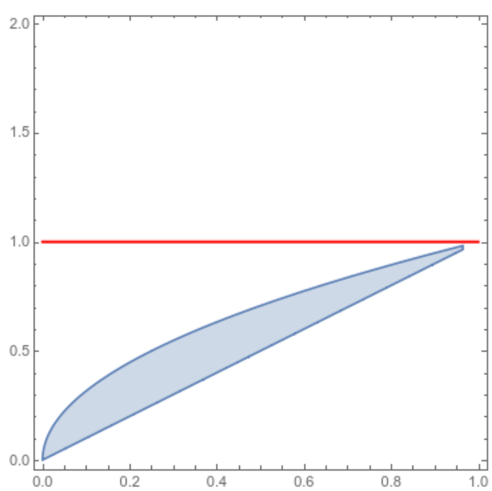
(b) Lightly try to sketch what the solid of revolution will look like. Can you describe it in words?

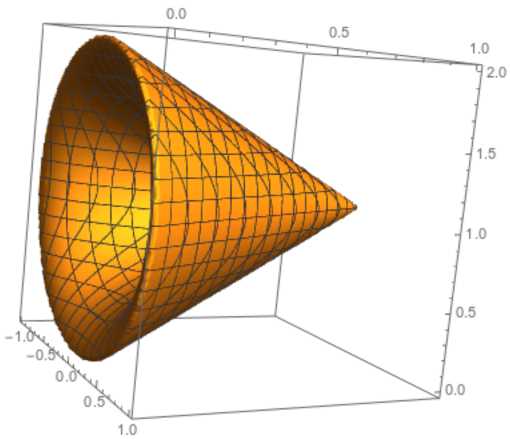
(c) Sketch in the slices you're going to use. Write down a formula for the volume of one slice. (This should involve a dx).

(d) Set up an integral that computes the volume of the whole solid.

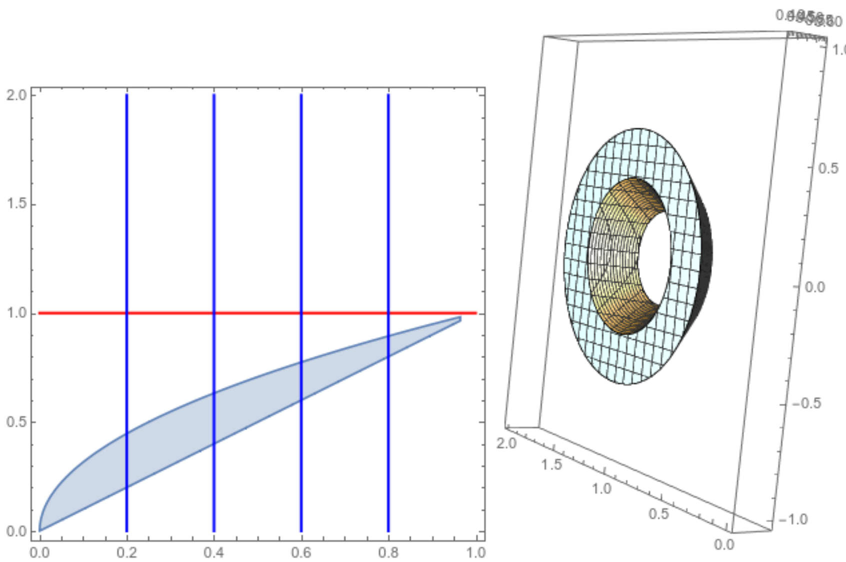
(e) Compute the volume.

Solution:

(a) 

(b) 

It sort of looks like a cone partially hollowed out on the inside. You can see it as a stack of progressively smaller rings.

(c) 

Each slice is roughly an annulus, with outer radius $1 - x$ and inner radius $1 - \sqrt{x}$. Thus the volume of a slice is $\pi(1-x)^2\,dx - \pi(1-\sqrt{x})^2\,dx$.

(d)

$$V = \pi \int_0^1 (1-x)^2 - (1-\sqrt{x})^2\,dx = \pi \int_0^1 x^2 - 3x + 2\sqrt{x}\,dx.$$

(e)

$$V = \pi \int_0^1 (1-x)^2 - (1-\sqrt{x})^2\,dx = \pi \int_0^1 x^2 - 3x + 2\sqrt{x}\,dx$$
$$= \pi \left(\frac{x^3}{3} - \frac{3x^2}{2} + \frac{4}{3}x^{3/2} \right)\Bigg|_0^1 = \pi \left(\frac{1}{3} - \frac{3}{2} + \frac{4}{3} \right) = \frac{\pi}{6}.$$

Problem 4. We want to find the volume of the solid obtained by rotating the region bounded by $xy = 1, x = 0, y = 1, y = 3$ about the x-axis.

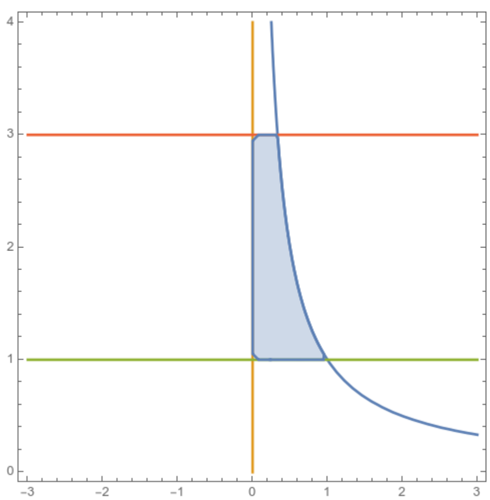
(a) Sketch a picture of this region. Try to visualize what the solid of revolution will look like.

(b) Try setting up an integral using the washer method. Are we integrating dx or dy? What's annoying about this setup?

(c) Now let's set up an integral using the shells method. Here are we integrating dx or dy? What's harder about this? What's easier?

(d) Compute one of those integrals.

Solution:

(a) 

(b) To use washers, we have to integrate dx, which isn't bad. But we do have to break the shape up into two pieces. For $0 \leq x \leq 1/3$ we have a shape from $y = 1$ to $y = 3$; for $1/3 \leq x \leq 1$ we have a shape from $y = 1$ to $y = 1/x$. So our integral is

$$V = \pi \int_0^{1/3} 3^2 - 1^2 \, dx + \pi \int_{1/3}^{1} \left(\frac{1}{x}\right)^2 - 1^2 \, dx.$$

(c) If we integrate by cylindrical shells, we have to integrate dy. We have y varying from 1 to 3, and the "height" of each cylinder is $1/y - 0$. So the volume is

$$V = \int_1^3 2\pi y(1/y) \, dy.$$

This is much easier because there's only one integral. (And the integral itself is pretty easy.) It's more annoying because we integrate dy, which is a bit more uncomfortable—and also because this is a newer technique.

(d) We can try to compute

$$\begin{aligned} V &= \pi \int_0^{1/3} 3^2 - 1^2 \, dx + \pi \int_{1/3}^1 \left(\frac{1}{x}\right)^2 - 1^2 \, dx \\ &= \pi 8x \Big|_0^{1/3} + \pi \left(\frac{-1}{x} - x\right)\Big|_{1/3}^1 \\ &= \pi \frac{8}{3} + \pi \left(-2\right) - \pi \left(-3 - 1/3\right) \\ &= \pi \frac{8}{3} - 2\pi + \frac{10}{3}\pi = 4\pi. \end{aligned}$$

But it's maybe easier to compute

$$\begin{aligned} V &= \int_1^3 2\pi y (1/y) \, dy \\ &= \int_1^3 2\pi \, dy \\ &= 2\pi y|_1^3 = 4\pi. \end{aligned}$$