Math 1231-13: Single-Variable Calculus 1
George Washington University Spring 2024
Recitation 3

Jay Daigle

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Problem 1. We want to compute $\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3}$.

(a) What rule do we know we need to invoke here?

(b) What θ are we going to need to pick for this to work out, and why?

(c) Do algebra so that you can invoke the small angle approximation. What is the limit? (Are you using the AIF property?)

(d) Go back to the beginning, and see what our heuristic idea that $\sin(\theta) \approx \theta$ would have told you. Does that match with what you got?

Solution:

(a) We need to use the small angle approximation, because this problem requires us to make trig interact with algebra.

(b) We basically have to take $\theta = x^2 - 9$, because that's what's inside the sin.

(c) We need to get a θ on the bottom as well. So we take

$$\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} =^{AIF} \lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} \cdot \frac{x+3}{x+3} = \lim_{x\to 3} \frac{\sin(x^2-9)(x+3)}{x^2-9}$$
$$= \lim_{x\to 3} \frac{\sin(x^2-9)}{x^2-9} \cdot \lim_{x\to 3} x+3 = 1 \cdot (3+3) = 6.$$

You have to use AIF in that first equality, because you're making the function undefined at -3.

There are a couple different ways to make this algebra work out. You might also try

$$\begin{aligned}\lim_{x\to 3}\frac{\sin(x^2-9)}{x-3} &=^{AIF} \lim_{x\to 3}\frac{\sin(x^2-9)/x^2-9\cdot x^2-9}{x-3}\\ &= \lim_{x\to 3}\frac{x^2-9}{x-3} =^{AIF} \lim_{x\to 3} x+3 = 6,\end{aligned}$$

which is essentially the same logic.

(d)

We have a sin(0) on the top and a 0 on the bottom, but the 0s don't come from the same form; we need to get a x^2-9 term on the bottom. Multiplication by the conjugate gives

$$\begin{aligned}\lim_{x\to 3}\frac{\sin(x^2-9)}{x-3} &= \lim_{x\to 3}\frac{\sin(x^2-9)}{x-3}\cdot\frac{x+3}{x+3} = \lim_{x\to 3}\frac{\sin(x^2-9)(x+3)}{x^2-9}\\ &= \lim_{x\to 3}\frac{\sin(x^2-9)}{x^2-9}\cdot\lim_{x\to 3} x+3 = 1\cdot(3+3) = 6.\end{aligned}$$

Problem 2. We want to think about the ways that infinity doesn't really work like a number, and we can't do arithmetic with it.

(a) To start: what is $\lim_{x\to 0} 1/x$, and why?

(b) Let's look at $\lim_{x\to 0} 1/x + 1/x$. If we computed the limit of each fraction individually, what indeterminate form would we get?

(c) How do we actually compute $\lim_{x\to 0}\frac{1}{x}+\frac{1}{x}$? (Hint: combine them into one fraction.) Does this make sense in light of what you got in part (b)?

(d) Now consider $\lim_{x\to 0}\frac{1}{x}+\frac{x-1}{x-x^2}$. What is the limit of each piece, and what indeterminate form is this?

(e) Compute $\lim_{x\to 0}\frac{1}{x}+\frac{x-1}{x-x^2}$ directly. Does this make sense in light of what you got in part (d)?

(f) Now consider $\lim_{x\to 0} 1/x+1/x^2$. What indeterminate form would this represent? What is the limit? Do those make sense together?

(g) Finally, let's look at $\lim_{x\to 0}\frac{1}{x}+\frac{x^2-3x+2}{x^2-2x}$. What indeterminate form is this? What is the limit?

(h) What pattern do you see from all of these?

Solution:

(a) $\lim_{x\to 0} \frac{1^{\nearrow 1}}{x_{\searrow 0}} = \pm\infty$.

(b) This looks like $\infty + \infty$ as an indeterminate form.

(c) We see $\lim_{x\to 0} \frac{1}{x} + \frac{1}{x} = \lim_{x\to 0} \frac{2^{\nearrow 2}}{x_{\searrow 0}} = \pm\infty$. This seems to make sense; $\infty + \infty = \infty$ is perfectly reasonable.

(d) We already know $\lim_{x\to 0} \frac{1}{x} = \pm\infty$. We can compute that

$$\lim_{x\to 0} \frac{x-1^{\nearrow -1}}{x-x^2{}_{\searrow 0}} = \pm\infty.$$

So this is again $\infty + \infty$.

(e) By combining fractions, we get

$$\lim_{x\to 0} \frac{1}{x} + \frac{x-1}{x-x^2} = \lim_{x\to 0} \frac{1-x}{x-x^2} + \frac{x-1}{x-x^2} = \lim_{x\to 0} 0 = 0.$$

So here $\infty + \infty = 0$.

(f) We have a $\pm\infty$ plus a $+\infty$, so we get $\infty + \infty$ again. When we combine them into one term we get

$$\lim_{x\to 0} \frac{1}{x} + \frac{1}{x^2} = \lim_{x\to 0} \frac{x+1^{\nearrow 1}}{x^2{}_{\searrow 0}} = +\infty$$

since the denominator is $x^2 \geq 0$. So here $\infty + \infty = +\infty$.

We could heuristically say that $\frac{1}{x^2}$ goes to $+\infty$ "faster" than $\frac{1}{x}$ goes to $\pm\infty$, and so it wins out; but this is really vague and handwavy so we try to replace it with more precise arguments like this one.

(g) We compute $\lim_{x\to 0} \frac{x-3x+2^{\nearrow 2}}{x^2-2x_{\searrow 0}} = \pm\infty$, so this is, again, $\infty + \infty$. The actual limit is

$$\lim_{x\to 0} \frac{1}{x} + \frac{x^2-3x+2}{x^2-2x} = \lim_{x\to 0} \frac{x-2+x^2-3x+2}{x^2-2x} = \lim_{x\to 0} \frac{x^2-2x}{x^2-2x} = \lim_{x\to 0} 1 = 1.$$

So here $\infty + \infty = 1$.

(h) In conclusion, if you know something looks like $\infty + \infty$, you don't really know anything about it at all.

Problem 3. (a) Consider $\lim_{x\to -\infty} \frac{x}{x+1}$. Can you come up with a heuristic guess about what this limit is?

(b) Can you carefully justify your guess from part (a).

(c) Now consider $\lim_{x\to+\infty} \frac{x}{\sqrt{3x^2+x}}$, and come up with a heuristic estimate for the limit.

(d) Carefully justify your guess from part (c).

(e) How would either of those calculations change if we take the limit to the other infinity?

Solution:

(a) If x is large, x and $x+1$ should behave basically the same; the 1 is insignificant compared to the x. So this limit should be 1.

(b)
$$\lim_{x\to-\infty} \frac{x}{x+1} = \lim_{x\to-\infty} \frac{1}{1+\frac{1}{x}} = \lim_{x\to-\infty} \frac{1}{1} = 1.$$

(c) When x is large, the x will be really large, but really small relative to the $3x^2$. So this should look like $\frac{x}{\sqrt{3x^2}}$ which goes to $\frac{1}{\sqrt{3}}$.

(d)
$$\begin{aligned}\lim_{x\to+\infty} \frac{x}{\sqrt{3x^2+1}} &= \lim_{x\to+\infty} \frac{1}{\sqrt{3x^2+1}/x} \\ &= \lim_{x\to+\infty} \frac{1}{\sqrt{3x^2+1}/\sqrt{x^2}} \\ &= \lim_{x\to+\infty} \frac{1}{\sqrt{3+\frac{1}{x^2}}} = \frac{1}{\sqrt{3}}.\end{aligned}$$

(e) The first wouldn't change at all. The second would change, because if $x > 0$ then $x = \sqrt{x^2}$, but if $x < 0$ then $x = -\sqrt{x^2}$. So we instead get
$$\begin{aligned}\lim_{x\to-\infty} \frac{x}{\sqrt{3x^2+1}} &= \lim_{x\to-\infty} \frac{1}{\sqrt{3x^2+1}/x} \\ &= \lim_{x\to-\infty} \frac{1}{\sqrt{3x^2+1}/-\sqrt{x^2}} \\ &= \lim_{x\to-\infty} \frac{-1}{\sqrt{3+\frac{1}{x^2}}} = \frac{-1}{\sqrt{3}}.\end{aligned}$$

Problem 4.

(a) We want to compute $\lim_{x\to+\infty} \sqrt{x^2+1} - x$. Can we just plug in here, or is this an indeterminate form? Why?

(b) When we have an indeterminate form, we generally want to write it as a big fraction, simplify, and factor. How can we do that here? We have to use a technique from last week to really get this to work.

(c) Once you have a big fraction, use it to compute the limit.

(d) How does this argument change if instead we want $\lim_{x\to+\infty}\sqrt{x^2+x+1}-x$?

(e) What is $\lim_{x\to+\infty}\sqrt{x^2+ax+1}-x$?

(f) What does the answer in part (e) say about $\lim_{x\to+\infty}\sqrt{x^2+2x+1}-x$? Why should the answer to this question be obvious?

Solution:

(a) This is indeterminate, of the form $\infty-\infty$.

(b) The simplest thing we could do is just write

$$\lim_{x\to+\infty}\sqrt{x^2+1}-x=\lim_{x\to+\infty}\frac{\sqrt{x^2+1}-x}{1}.$$

But that doesn't get us very far. Since we have a difference of square roots, we want to multiply by the conjugate.

(c) We get

$$\begin{aligned}\lim_{x\to+\infty}\sqrt{x^2+1}-x&=\lim_{x\to+\infty}\left(\sqrt{x^2+1}-x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\\&=\lim_{x\to+\infty}\frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x}=\lim_{x\to+\infty}\frac{1}{\sqrt{x^2+1}+x}\\&=\lim_{x\to+\infty}\frac{1/x}{\sqrt{1+1/x^2}+1}=0.\end{aligned}$$

(d)

$$\begin{aligned}\lim_{x\to+\infty}\sqrt{x^2+x+1}-x&=\lim_{x\to+\infty}\left(\sqrt{x^2+x+1}-x\right)\frac{\sqrt{x^2+x+1}+x}{\sqrt{x^2+x+1}+x}\\&=\lim_{x\to+\infty}\frac{x^2+x+1-x^2}{\sqrt{x^2+x+1}+x}=\lim_{x\to+\infty}\frac{x+1}{\sqrt{x^2+x+1}+x}\\&=\lim_{x\to+\infty}\frac{1+1/x}{\sqrt{1+1/x+1/x^2}+1}=\frac{1}{2}.\end{aligned}$$

(e) We can make essentially the same argument:

$$\begin{aligned}
\lim_{x\to+\infty} \sqrt{x^2+ax+1} - x &= \lim_{x\to+\infty} \left(\sqrt{x^2+ax+1} - x\right) \frac{\sqrt{x^2+ax+1}+x}{\sqrt{x^2+ax+1}+x} \\
&= \lim_{x\to+\infty} \frac{x^2+ax+1-x^2}{\sqrt{x^2+ax+1}+x} = \lim_{x\to+\infty} \frac{ax+1}{\sqrt{x^2+ax+1}+x} \\
&= \lim_{x\to+\infty} \frac{a+1/x}{\sqrt{1+a/x+1/x^2}+1} = \frac{a}{2}.
\end{aligned}$$

(f) By the answer from part (b), $\lim_{x\to+\infty} \sqrt{x^2+2x+1} - x = 2/2 = 1$. But we could also just observe that $x^2+2x+1 = (x+1)^2$, so

$$\lim_{x\to+\infty} \sqrt{x^2+2x+1} - x = \lim_{x\to+\infty} (x+1) - x = 1.$$