Math 1231 Spring 2024
Single-Variable Calculus I Section 11
Mastery Quiz 5
Due Tuesday, February 20

This week's mastery quiz has four topics. Everyone should submit topics M2 and S3. If you already have a 4/4 on M1 or a 2/2 on S2 (check Blackboard!) you don't need to submit them again.

Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Thursday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Computing Limits
- Major Topic 2: Computing Derivatives
- Secondary Topic 2: Definition of Derivative
- Secondary Topic 3: Linear Approximation

Name:

Recitation Section:

Major Topic 1: Computing Limits

(a) $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{x^2+x+1}}{x+3} =$

Solution:

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+x+1}}{x+3} = \lim_{x\to-\infty}\frac{-\sqrt{1+1/x+1/x^2}}{1+3/x} = \frac{-\sqrt{1+0+0}}{1+0} = -1.$$

(b) $\displaystyle\lim_{x\to3}\frac{1}{x-3}-\frac{3}{x^2-3x} =$

Solution:

$$\begin{aligned}\lim_{x\to3}\frac{1}{x-3}+\frac{3}{x^2-3x} &= \lim_{x\to3}\frac{x^2-3x-3(x-3)}{(x-3)(x^2-3x)}\\ &= \lim_{x\to3}\frac{x^2-6x+9}{x(x-3)^2}\\ &= \lim_{x\to3}\frac{1}{x} = \frac{1}{3}.\end{aligned}$$

(c) $\displaystyle\lim_{x\to4}\frac{\sqrt{5-x}-1}{x-4} =$

Solution:

$$\begin{aligned}\lim_{x\to4}\frac{\sqrt{5-x}-1}{x-4} &= \lim_{x\to4}\frac{5-x-1}{(x-4)(\sqrt{5-x}+1)}\\ &= \lim_{x\to4}\frac{4-x}{(x-4)(\sqrt{5-x}+1)}\\ &= \lim_{x\to4}\frac{-1}{\sqrt{5-x}+1} = \frac{-1}{2}.\end{aligned}$$

Major Topic 2: Computing Derivatives

(a) Compute $\displaystyle\frac{d}{dx}\sec\left(\frac{x^2+1}{\sqrt{x^3-2}}\right) =$

Solution:

$$\sec\left(\frac{x^2+1}{\sqrt{x^3-2}}\right)\tan\left(\frac{x^2+1}{\sqrt{x^3-2}}\right)\frac{2x\sqrt{x^3-2}-(x^2+1)\frac{1}{2}(x^3-2)^{-1/2}\cdot 3x^2}{x^3-2}.$$

(b) $\displaystyle\frac{d}{dx}\sec(\tan(\cos((x+1)^2)))$

Solution:

$$\sec(\tan(\cos((x+1)^2)))\tan(\tan(\cos((x+1)^2)))\sec^2(\cos((x+1)^2))(-\sin((x+1)^2))2(x+1)$$

Secondary Topic 2: Definition of Derivative

(a) If $f(x) = \sqrt{x+3}$, find $f'(6)$, directly from the definition of derivative.

Solution:

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(6+h) - f(6)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{9+h} - 3}{h} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{9+h}+3)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3} \\ &= \lim_{h\to 0} \frac{1}{6}. \end{aligned}$$

If $g(x) = x^3 - 3x$, find $g'(x)$, directly from the definition of derivative.

Solution:

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 - 3(x+h) - x^3 + 3x}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h - x^3 + 3x}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + h^2 - 3 = 3x^2 - 3. \end{aligned}$$

Secondary Topic 3: Linear Approximation

(a) Find a linear approximation to the function $f(x) = \frac{x^3}{1+x}$ near the point $a = 1$ and use it to approximate $f(1.3)$.

Solution:

$$
\begin{aligned}
f(1) &= \frac{1}{2} \\
f'(x) &= \frac{3x^2(1+x) - x^3}{(1+x)^2} \\
f'(1) &= \frac{6-1}{4} = \frac{5}{4} \\
f(x) &\approx \frac{1}{2} + \frac{5}{4}(x-1) \\
f(1.3) &\approx \frac{1}{2} + \frac{5}{4} \cdot .3 = \frac{20}{40} + \frac{15}{40} = \frac{35}{40} = \frac{7}{8}.
\end{aligned}
$$

(b) Give a formula for a linear approximation of $f(x) = x\sqrt{x+1}$ near the point $a = 3$. Use your answer to estimate $f(2.8)$.

Solution:

$$
\begin{aligned}
&f'(x) = \sqrt{x+1} + \frac{x}{2\sqrt{x+1}} \\
&f'(3) = 2 + \frac{3}{4} = \frac{11}{4} \\
&f(x) \approx f(a) + f'(a)(x-a) = 6 + \frac{11}{4}(x-3).
\end{aligned}
$$

$f(2.8) \approx 6 + \frac{11}{4}(-.2) = 6 - \frac{11}{20} = \frac{109}{20}$.