Math 1231-13: Single-Variable Calculus 1
George Washington University Spring 2024
Recitation 4

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Problem 1. (a) Let $h(x) = \tan^2(x)$. Find functions f and g so that $h(x) = (f \circ g)(x)$.

(b) Compute $f'(x)$ and $g'(x)$. Use that info to compute $h'(x)$.

(c) Now let $h(x) = \tan(x^2)$. Find functions f and g so that $h(x) = (f \circ g)(x)$.

(d) Compute $f'(x)$ and $g'(x)$. Use that information to compute $h'(x)$.

Solution:

(a) We can take $f(x) = x^2$ and $g(x) = \tan(x)$.

(b) $f'(x) = 2x$ and $g'(x) = \sec^2(x)$, so

$$h'(x) = f'(g(x)) \cdot g'(x) = f'(\tan(x)) \cdot g'(x) = 2\tan(x) \cdot \sec^2(x).$$

(c) Now we have $f(x) = \tan(x)$ and $g(x) = x^2$.

(d) Now we have $f'(x) = \sec^2(x)$ and $g'(x) = 2x$, so

$$h'(x) = f'(g(x)) \cdot g'(x) = f'(x^2) \cdot g'(x) = \sec^2(x^2) \cdot 2x.$$

Problem 2. Consider the function $\sec^2(x^2 + 1)$

(a) Find functions f and g so that $(f \circ g)(x) = \sec^2(x^2 + 1)$.

(b) Talk to the people next to you. Did they pick the same f and g that you did? Can you find a different pair of functions f and g that also work?

(c) Find functions f, g, h so that $(f \circ g \circ h)(x) = \sec^2(x^2+1)$.

(d) Compute f', g', and h'.

(e) What is $\frac{d}{dx}\sec^2(x^2+1)$?

Solution:

(a) There are basically two choices here. You could say that $f(x) = \sec^2(x)$ and $g(x) = x^2+1$, which is maybe the more obvious choice; or you could say that $f(x) = x^2$ and $g(x) = \sec(x^2+1)$.

(b) This is really a composite of three functions, which is why you could make different choices here.

(c) $f(x) = x^2, g(x) = \sec(x), h(x) = x^2+1$. (Technically there are other things you could do, like $g(x) = \sec(x+1)$ and $h(x) = x^2$, but those are moderately silly.)

(d) $f'(x) = 2x, g'(x) = \sec(x)\tan(x), h'(x) = 2x$.

(e)

$$\begin{aligned}\frac{d}{dx}\sec^2\left(x^2+1\right) &= f'(g(h(x)) \cdot g'(h(x)) \cdot h'(x) \\ &= f'(\sec(x^2+1)) \cdot g'(x^2+1) \cdot h'(x) \\ &= 2\sec(x^2+1) \cdot \sec(x^2+1)\tan(x^2+1) \cdot 2x.\end{aligned}$$

Problem 3. Find

$$\frac{d}{dx}\frac{\sin(x^2)+\sin^2(x)}{x^2+1}$$

Solution:

$$\begin{aligned}\frac{d}{dx}\frac{\sin(x^2)+\sin^2(x)}{x^2+1} &= \frac{(\sin(x^2)+\sin^2(x))'(x^2+1) - 2x(\sin(x^2)+\sin^2(x))}{(x^2+1)^2} \\ &= \frac{(\cos(x^2)\cdot 2x + 2\sin(x)\cos(x))(x^2+1) - 2x(\sin(x^2)+\sin^2(x))}{(x^2+1)^2}.\end{aligned}$$

Problem 4. (a) Compute

$$\frac{d}{dx}\sqrt{\frac{\sqrt{x}+1}{(\cos x+1)^2}}$$

Solution:

$$\frac{d}{dx}\sqrt{\frac{\sqrt{x}+1}{(\cos x+1)^2}} = \frac{1}{2}\left(\frac{\sqrt{x}+1}{(\cos x+1)^2}\right)^{-1/2} \cdot \left(\frac{\sqrt{x}+1}{(\cos x+1)^2}\right)'$$
$$= \frac{1}{2}\left(\frac{\sqrt{x}+1}{(\cos x+1)^2}\right)^{-1/2} \cdot \frac{\frac{1}{2}x^{-1/2}(\cos x+1)^2 - 2(\cos x+1)(-\sin x)(\sqrt{x}+1)}{(\cos x+1)^4}$$

(b) Find

$$\frac{d}{dx}\tan^4(\sqrt[3]{x^5+x^3+2}+1).$$

Solution:

$$\frac{d}{dx}\tan^4(\sqrt[3]{x^5+x^3+2}+1) = 4\tan^3(\sqrt[3]{x^5+x^3+2}+1)\cdot\sec(\sqrt[3]{x^5+x^3+2}+1)$$
$$\cdot\tan(\sqrt[3]{x^5+x^3+2}+1)\cdot(\sqrt[3]{x^5+x^3+2}+1)'$$
$$= 4\tan^4(\sqrt[3]{x^5+x^3+2}+1)\sec(\sqrt[3]{x^5+x^3+2}+1)$$
$$\cdot\left(\frac{1}{3}(x^5+x^3+1)^{-2/3}\cdot(5x^4+3x^2)\right).$$

Problem 5 (Bonus)**.** Calculate

$$\frac{d}{dx}\left(\frac{\sin^2\left(\frac{x^2+1}{\sqrt{x-1}}\right)+\sqrt{x^3-2}}{\cos(\sqrt{x^2+1}+1)-\tan(x^4+3)}\right)^{5/3}$$

Problem 6 (Geometric Series)**.** Another function it's sometimes important to approximate is the "geometric series" formula $f(x) = \frac{1}{1-x}$, near $x = 0$.

(a) What is $f'(x)$?

(b) Find a linear approximation for $f(x)$ near $x = 0$.

(c) Use this formula to estimate $\frac{1}{.9}$ and $\frac{1}{1.01}$. Do these answers make sense?

(d) Use your formula to estimate $\frac{1}{1.5}$ and $frac10.5$. Do these answers make sense?

(e) Use your formula to estimate $f(-1)$ and $f(1)$. Do these answers make sense?

Solution:

(a) $f'(x) = -(1-x)^{-2} = \frac{1}{(1-x)^2}$. This is tricky; you get a negative sign from the power rule, but another from the chain rule that cancels it out.

(This is a weird way to write the function! Why not just use $\frac{1}{1+x}$? Because this setup makes more sense in a lot of the applications people want to use it for. You'll see why when you study power series in Calculus 2.)

(b) $f'(0) = 1$, so our linear approximation is $f(x) \approx 1 + x$.

(c) $\frac{1}{.9} = f(.1) \approx 1.1$. The true answer is $1.\overline{11}$, so that checks out.

$\frac{1}{1.01} = f(-.01) \approx .99$. The true answer is $.990099\ldots$, which also makes sense.

(d) $\frac{1}{1.5} = f(-0.5) \approx 0.5$. The true answer is $2/3 \approx .\overline{66}$ so this is, like, okay-ish.

$\frac{1}{0.5} = f(0.5) \approx 1.5$. The true answer is 2, so this is again okay, but not great.

(e) $f(-1) \approx 0$. But $f(-1) = 1/2$, so that doesn't make a ton of sense. This is because (-1) is "far away" from zero for our purposes. And how do we know it's far away? Well...

$f(1) \approx 0$. But $f(1)$ is utterly undefined, since it asks us to divide by 0. We've gone too far away for the linear approximation to work at all.