Math 1231 Spring 2024
Single-Variable Calculus I Section 11
Mastery Quiz 7
Due Tuesday, March 5

This week's mastery quiz has four topics. Everyone should submit topics S5 and S6. If you already have a 4/4 on M2 or a 2/2 on S4 (check Blackboard!) you don't need to submit them again.

Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Thursday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 2: Computing Derivatives
- Secondary Topic 4: Rates of Change
- Secondary Topic 5: Implicit Differentiation
- Secondary Topic 6: Related Rates

Name:

Recitation Section:

Major Topic 2: Computing Derivatives

(a) Compute $\frac{d}{dx}\tan^{3/5}\left(\sec\left(x^3-4\right)\right)$

Solution:

$$\frac{3}{5}\tan^{-2/5}\left(\sec\left(x^3-4\right)\right)\sec^2\left(\sec\left(x^3-4\right)\right) \cdot \sec\left(x^3-4\right)\tan\left(x^3-4\right)3x^2$$

(b) Compute $\frac{d}{dt}\sqrt[5]{\frac{\tan^2(t^2+1)+2}{\sin(2t)-2t}}$.

Solution:

$$\frac{d}{dt}\sqrt[5]{\frac{\tan^2(t^2+1)+2}{\sin(2t)-2t}} = \frac{1}{5}\left(\frac{\tan^2(t^2+1)+2}{\sin(2t)-2t}\right)^{-4/5} \cdot \frac{\left(2\tan(t^2+1)\sec^2(t^2+1)2t\right)\left(\sin(2t)-2t\right)-\left(2\cos(2t)-2\right)\left(\tan^2(t^2+1)+2\right)}{\left(\sin(2t)-2t\right)^2}.$$

Secondary Topic 4: Rates of Change

(a) Suppose you are running a factory that makes industrial equipment, and if you manufacture m machines in a day you make a profit of $P(m) = 1000m - m^3$ dollars.

  (a) What are the units of $P'(m)$? What does it mean if $P'(m)$ is positive?

  (b) Calculate $P'(10)$. What does this calculation tell you physically? What observation could you make to check this calculation?

Solution:

  (i) $P'(m)$ has units of dollars per machine. If $P'(m)$ is positive, making an additional machine each day will increase your profit.

  (ii) $P'(m) = 1000-3m^2$ so $P'(10) = 1000-300 = 700$. This means if you're currently making ten machines per day, you should get an extra \$700 in profit if you start making eleven machines per day.

  In economics they call this your marginal revenue.

(b) Suppose the distance between two particles in centimeters is given as a function of time in seconds by the formula $d(t) = t^3 + 4t^2 + 5t + 4$.

  (i) When is the velocity zero?

  (ii) When is the acceleration zero?

Solution:

(i) $d'(t) = 3t^2 + 8t + 5 = (3t+5)(t+1)$ so the velocity is zero when $t = -1, -5/3$.

(ii) $d''(t) = 6t + 8$ is zero when $t = -4/3$.

Secondary Topic 5: Implicit Differentiation

(a) Find an equation for the line tangent to the curve $x^2y - xy^3 = xy + 3$ at the point $(3, 1)$.

Solution:

$$\begin{aligned} 2xy + x^2y' - y^3 - 3xy^2y' &= y + xy' \\ 6 + 9y' - 1 - 9y' &= 1 + 3y' \\ 4 &= 3y' \\ y' &= 4/3 \end{aligned}$$

and thus an equation for the tangent line is

$$y - 1 = \frac{4}{3}(x - 3).$$

(b) Find a formula for $\frac{d^2y}{dx^2}$ in terms of x and y if $\sin(xy) = x + y$.

Solution: Using implicit differentiation, we have

$$\begin{aligned} \cos(xy)(y + xy') &= 1 + y' \\ y\cos(xy) - 1 &= (1 - \cos(xy)x)y' \end{aligned}$$

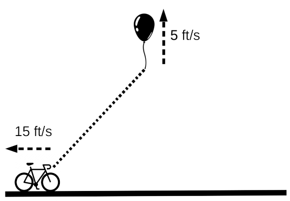
$$\begin{aligned} y' &= (\frac{y\cos(xy) - 1}{1 - \cos(xy)x} \\ y'' &= \frac{(y'\cos(xy) - y\sin(xy)(y + xy'))\,(1 - \cos(xy)x) - (y\cos(xy) - 1)(x\sin(xy)(y + xy') - \cos(xy))}{(1 - \cos(xy)x)^2} \\ &= \frac{\left(\left(\frac{y\cos(xy)-1}{1-\cos(xy)x}\right)\cos(xy) - y\sin(xy)(y + x\left(\frac{y\cos(xy)-1}{1-\cos(xy)x}\right))\right)(1 - \cos(xy)x) - (y\cos(xy) - 1)(x\sin(xy}{(1 - \cos(xy)x)^2} \end{aligned}$$

Secondary Topic 6: Related Rates

A balloon is rising at a constant speed of 5 feet per second. A boy is cycling along a straight road at a speed of 15 feet per second. When he passes under the balloon, it is 45 feet above him. We want to know how fast is he distance between the boy and the balloon is increasing 3 seconds later.

(a) Choose an equation to use for this problem, and explain why you chose that equation.

(b) Use calculus to answer the question. Make sure you answer with a complete sentence.



Solution: We know one distance, and how fast it's changing. We want to know how fast another distance is changing, so the Pythagorean theorem, which relates distances to each other, seems like a reasonable choice. So we use the equation $d^2 = w^2 + h^2$.

We see that the height of the balloon is $h = 60\text{ft}$, and the derivative is $h' = 5\text{ft/s}$. The distance between the boy and the point under the balloon is $w = 45\text{ft}$ and the derivative is $w' = 15\text{ft/s}$. The distance between them is given by $d^2 = w^2 + h^2$, and so we can compute first that the current distance is 75 feet, and then that

$$\begin{aligned} 2dd' &= 2ww' + 2hh' \\ dd' &= ww' + hh' \\ 75\text{ft}\,d' &= 45\text{ft} \cdot 15\text{ft/s} + 60\text{ft} \cdot 5\text{ft/s} = 675\text{ft}^2/\text{s} + 300\text{ft}^2/\text{s} = 975\text{ft}^2/\text{s} \\ d' &= \frac{975}{75}\text{ft/s} = \frac{325}{25}\text{ft/s} = 13\text{ft/s}. \end{aligned}$$

Thus the distance between the boy and the balloon is increasing by 13 feet per second.