Math 1231-13: Single-Variable Calculus 1
George Washington University   Spring 2024
Recitation 7

Jay Daigle

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Problem 1. A twenty foot ladder rests against a wall. The bit on the wall is sliding down at 1 foot per second. How quickly is the bottom end moving when the top is 12 feet from the ground?

(a) Draw a picture of this situation.

(b) What is the question you're trying to answer? What do you expect it to look like? Should it be positive or negative? What units do you expect?

(c) What equation should we use here, and why?

(d) Use a derivative to calculate the answer to the question. Does your answer make sense?

Solution: Let h be the height of the ladder on the wall, and b be the distance of the foot of the ladder from the wall. Then $h = 12, h' = -1$, and $b = \sqrt{400 - 144} = 16$. We have

$$\begin{aligned} h^2 + b^2 &= 400 \\ 2hh' + 2bb' &= 0 \\ 2 \cdot 12 \cdot (-1) + 2 \cdot 16 \cdot b' &= 0 \\ b' &= \frac{24}{32} = 3/4 \end{aligned}$$

so the foot of the ladder is sliding away from the wall at 3/4 ft/s. Again, the direction of the sliding is correct (away from the wall), and the number seems plausible.

Problem 2. A rectangle is getting longer by one inch per second and wider by two inches per second. When the rectangle is 5 inches long and 7 inches wide, how quickly is the area increasing?

Bonus Draw a picture of this situation.

Bonus What is the question you're trying to answer? What do you expect it to look like? Should it be positive or negative? What units do you expect?

Bonus What equation should we use here, and why?

Bonus Use a derivative to calculate the answer to the question. Does your answer make sense?

Bonus To check things: how long and wide will the rectangle be after one inch? How much will the area have increased? Does that make sense with your answer to the related rates problem?

Bonus Bonus: where have we seen basically this argument before?

Solution:

(a) It's a rectangle.

(b) We want to know how quickly the area is increasing, so we're looking for $\frac{dA}{dt}$, and the units should be in^2/s.

(c) We can relate all our quantities with the formula for the area of a rectangle: $A = \ell w$ relates the area, which we want to know about, to the length and width, which we do know about.

We have $\ell = 5\text{in}, w = 7\text{in}, \frac{d\ell}{dt} = 1\text{in/s}, \frac{dw}{dt} = 2\text{in/s}$. Taking a derivative gives us

$$\begin{aligned} \frac{dA}{dt} &= \ell \frac{dw}{dt} + w \frac{d\ell}{dt} \\ &= 5\text{in} \cdot 2\text{in/s} + 7\text{in} \cdot 1\text{in/s} \\ &= 17\text{in}^2/\text{s}. \end{aligned}$$

The units are right (the rate at which area is changing per second), and the direction is right (the area should be increasing, and this derivative is positive). It's really hard to see if the size is right using our intuition; people in general have bad intuition for the rate at which area changes in response to lengths.

One second later, we'd have $\ell = 6$in and $w = 9$in for a total area of 54in^2. This is an increase of 19in^2 over our starting area of 35in^2, and 17 is a pretty good approximation of 19.

The derivative of the area formula is just the product rule; we saw basically this same picture during the proof of the product rule.

Problem 3. A spot light is on the ground 36 ft away from a wall and a 5 ft tall person is walking towards the wall at a rate of 4 ft/sec. How fast is the height of the shadow changing when the person is 24 feet from the wall? Is the shadow increasing or decreasing in height at this time?

(a) Draw a picture of this situation.

(b) What is the question you're trying to answer? What do you expect it to look like? Should it be positive or negative? What units do you expect?

(c) What equation should we use here, and why?

(d) Use a derivative to calculate the answer to the question. Does your answer make sense?

Solution:

(a)

(b) We want to know how fast the height of the shadow is changing. This should be in feet per second. Physically, it seems like the shadow should be shrinking.

(c) This is a similar triangles problem, because we can compare the triangle with height given by the person, to a triangle with height given by the shadow.

(d) Let h be the height of the shadow, and d be the distance between the wall and the person. Then we want to find h'. We currently have $d = 24$. We know by similar triangles that $\frac{36-d}{36} = \frac{5}{h}$, which tells us that currently $h = 15$.

Then we have $d' = -4$. We compute

$$\begin{aligned} \frac{-d'}{36} &= \frac{-5h'}{h^2} \\ \frac{1}{9} &= \frac{-h'}{45} \\ h' &= \frac{-45}{9} = -5. \end{aligned}$$

Thus the shadow's height is decreasing by 5 feet per second.

Problem 4. Consider the function $f(x) = x^3 - 3x^2 + 1$ on $[-1, 4]$.

(a) Does this function have absolute extrema? Why?

(b) What are the critical points of this function?

(c) How many absolute extrema are there? What are they, and where are they?

Solution:

(a) This function is continuous on a closed interval, so by the Extreme Value Theorem it must have an absolute maximum and an absolute minimum.

(b) $f'(x) = 3x^2 - 6x$ and is defined everywhere. We have $3x^2 - 6x = 0$ when $x = 0$ or $x = 2$, so the critical points are 0 and 2.

(c) We need to evaluate all the possible extrema: the critical poitns and the endpoints. We compute $f(-1) = -3, f(0) = 1, f(2) = -3, f(4) = 17$. Thus the absolute maximum is 17 at 4, and the absolute minimum is -3 at -1 and 2.

Problem 5. Let's find the global extrema of $g(x) = \sqrt[3]{x^3 + 6x^2}$ on the closed interval $[-5, 5]$.

(a) Does this function have absolute extrema? Why?

(b) What are the critical points of this function?

(c) How many absolute extrema are there? What are they, and where are they? (Hint: you may need to use a calculator at the last step.)

Solution:

(a) This function is continuous on a closed interval, so by the Extreme Value Theorem it must have an absolute maximum and an absolute minimum.

(b) We take the derivative, and compute

$$g'(x) = \frac{1}{3}(x^3 + 6x^2)^{-2/3}(3x^2 + 12x) = \frac{3x(x+4)}{3\sqrt[3]{(x^3 + 6x^2)^2}}.$$

This derivative is zero when $x = -4$, and it does't exist when $x = 0$ or $x = -6$. (You might think that $g'(0) = 0$, but it's actually just undefined.) The critical points are $-6, -4, 0$, but we can ignore the case where $x = -6$, since it's not inside the interval.

(c) We need to evaluate all the possible extrema: the critical points inside the interval, and the endpoints. We might need a calculator here, but we

$$\begin{aligned} g(-5) &= \sqrt[3]{-125+150} = \sqrt[3]{25} \approx 2.9 \\ g(-4) &= \sqrt[3]{-64+96} = \sqrt[3]{32} \approx 3.17 \\ g(0) &= \sqrt[3]{0} = 0 \\ g(5) &= \sqrt[3]{125+150} = \sqrt[3]{275} \approx 6.5. \end{aligned}$$

Thus the absolute minimum is 0, which occurs at 0, and the absolute maximum is $\sqrt[3]{275} \approx 6.5$, which occurs at 5.

Note that if you forget about the critical point where $g'(x)$ is undefined, you will miss the minimum! The minimum is very definitely not $\sqrt[3]{25}$, which you can see if you graph the function.