Math 1231 Spring 2024
Single-Variable Calculus I Section 11
Mastery Quiz 8
Due Tuesday, March 19

This week's mastery quiz has four topics. Everyone should submit topic M3. If you already have a 2/2 on S5 or S6 (check Blackboard!) you don't need to submit them again.

Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Thursday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Optimization and Extrema
- Secondary Topic 5: Implicit Differentiation
- Secondary Topic 6: Related Rates

Name:

Recitation Section:

Major Topic 3: Optimization and Extrema

(a) The function $g(x) = 3x^4 - 2x^3 - 3x^2 + 5$ has absolute extrema either on the interval $(-1, 2)$, or on the interval $[-1, 2]$. Pick one of those intervals, explain why g has extrema on that interval, and find the absolute extrema.

Solution: g is continuous on the closed interval $[-1, 2]$, so by the Extreme Value Theorem it has a maximum and a minimum on the interval. This must happen at a critical point or an endpoint. (This argument is necessary! Otherwise there's no reason to expect the largest local max to be a global max.)

We compute

$$g'(x) = 12x^3 - 6x^2 - 6x = 6x(2x^2 - x - 1) = 6x(2x + 1)(x - 1)$$

which is zero at $0, 1, -1/2$. Then we compute

$$\begin{aligned} g(-1) &= 7 \\ g(-1/2) &= \frac{3}{16} + \frac{1}{4} - \frac{3}{4} + 5 = 5 - \frac{5}{16} = \frac{75}{16} = 4.6875 \\ g(0) &= 5 \\ g(1) &= 3 \\ g(2) &= 48 - 16 - 12 + 5 = 25. \end{aligned}$$

Thus g has an absolute maximum of 25 at 2, and an absolute minimum of 3 at 1.

(b) Classify all the critical points and relative extrema of $h(x) = x^3/(x + 1)$. (For each critical point, tell me whether it is a relative maximum, a relative minimum, or neither.)

Solution: We have

$$h'(x) = \frac{3x^2(x + 1) - x^3}{(x + 1)^2} = \frac{3x^3 + 3x^2 - x^3}{(x + 1)^2} = \frac{x^2(2x + 3)}{(x + 1)^2}$$

The critical points are thus at 0 and at $-3/2$, and a fake one at -1. We make a chart:

	x^2	$2x + 3$	$(x + 1)^{-2}$	$h'(x)$
$x < -3/2$	+	−	+	−
$-3/2 < x < -1$	+	+	+	+
$-1 < x < 0$	+	+	+	+
$0 < x$	+	+	+	+

This tells us that we have a local minimum at $x = -3/2$, and no other extrema. We compute $h(-3/2) = -27/8/(-1/2) = 27/4$, so the sole local minimum is $(-3/2, 27/4)$.

Alternatively we could try the second derivative test. The second derivative is

$$h''(x) = \frac{2x(x^2 + 3x + 3)}{(x + 1)^3}.$$

We compute that

$$\begin{aligned} h''(0) &= 0 \\ h''(-3/2) &= \frac{-3(9/4 - 9/2 + 3)}{(-1/2)^3} = 24(3/4) = 18 > 0. \end{aligned}$$

This tells us we have a relative minimum at $-3/2$, but doesn't help us with the critical point at 0. Note: we *cannot* conclude from this that we don't have an extremum at 0, even though that is in fact true!

Secondary Topic 5: Implicit Differentiation

(a) Find an equation for the line tangent to the curve $3x^2y + 5xy^2 = 2x$ at the point $(1, -1)$.

Solution:

$$\begin{aligned} 6xy + 3x^2y' + 5y^2 + 10xyy' &= 2 \\ -6 + 3y' + 5 - 10y' &= 2 \\ -7y' &= 3 \\ y' &= -3/7 \end{aligned}$$

and thus an equation for the tangent line is

$$y + 1 = \frac{-3}{7}(x - 1).$$

(b) Find a formula for y' in terms of x and y if $\sqrt{x + y} = x^3y^2$.

Solution:

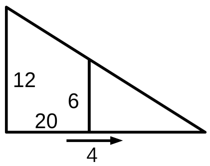
$$\begin{aligned} \frac{1}{2}(x+y)^{-1/2}(1 + y') &= 3x^2y^2 + 2x^3yy' \\ \frac{1}{2}(x+y)^{-1/2}y' - 2x^3yy' &= 3x^2y^2 - \frac{1}{2}(x+y)^{-1/2} \\ y' &= \frac{3x^2y^2 - \frac{1}{2}(x+y)^{-1/2}}{\frac{1}{2}(x+y)^{-1/2} - 2x^3y}. \end{aligned}$$

Secondary Topic 6: Related Rates

A street light is mounted at the top of a 12-foot-tall pole. A six-foot-tall man walks straight away from the pole at 4 feet per second. We want to know how fast the length of his shadow is changing when he is twenty feet from the pole.

(a) Choose an equation to use for this problem, and explain why you chose that equation.

(b) Use calculus to answer the question. Make sure you answer with a complete sentence.



Solution: After drawing a picture, we see we have two triangles in the same shape: we know how one triangle is changing, and we want to figure out how the other is changing, so we should relate those similar triangles.

Let d be the distance of the man from the pole. Then $d = 20$ and $d' = 4$. If s is the length of the shadow, then we have $s/6 = (d+s)/12$ so we get

$$\begin{aligned} s &= \frac{d+s}{2} \\ s' &= d'/2 + s'/2 \\ s'/2 &= d'/2 \\ s' &= d' = 4. \end{aligned}$$

Thus the length of the shadow is growing at 4 feet per second.