Math 1231-13: Single-Variable Calculus 1

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Recitation 8

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Problem 1. Let $f(x) = |x^3 - 3x|$ on the closed interval $[-1, 3]$.

(a) Does this function have absolute extrema? Why?

(b) What are the critical points of this function?

(c) How many absolute extrema are there? What are they, and where are they?

Solution:

(a) This function is continuous on a closed interval, so by the Extreme Value Theorem it must have extrema.

(b) Absolute value is funny here. Technically speaking, we know that

$$f'(x) = \frac{x^3 - 3x}{|x^3 - 3x|} \cdot (3x^2 - 3).$$

This is undefined when $0 = x^3 - 3x = x(x^2 - 3) = x(x - \sqrt{3})(x + \sqrt{3})$, which happens when $x = 0$ or $x = \pm\sqrt{3}$. It's zero when $3x^2 - 3 = 0$, which happens when $x = \pm 1$.

(c) We only care what happens in $[-1, 3]$, so we care about the critical points $-1, 0, 1, \sqrt{3}$, and also the endpoints -1 and 3. (But of course we don't have to look at -1 twice). We compute

$$f(-1) = 2 \qquad f(0) = 0 \qquad f(1) = 2$$
$$f(\sqrt{3}) = 0 \qquad f(3) = 18.$$

Thus we see the maximum value is 18, achieved at 3; and the minimum value is 0, achieved at 0 and $\sqrt{3}$.

The non-differentiable points were very important here. If we forgot about them, we'd only see $-1, 1, 3$, and conclude that the minimum value was 2.

Problem 2. We talked about using the combination of the Intermediate Value Theorem and Rolle's Theorem to figure out exactly how many solutions an equation has.

Let $g(x) = x^5 + x^3 + x - 1$. This is a function which has no roots you can actually write down in a useful way; it's a theorem that you can't give a nice algebraic description of the solutions to $x^5 + x^3 - 1 = 0$. But we can say some things about them.

(a) First, we want to get an idea of how many solutions we expect. Try plugging some small, easy numbers into this function, until you think you understand the function. How many solutions should it have?

(b) Now we want to prove we have to have a solution. Write up an argument in terms of the Intermediate Value Theorem to prove that $x^5 + x^3 - 1 = 0$ has a solution.

(c) Now we'll prove that there can't be any more. Check that this function satisfies the conditions of Rolle's Theorem. How does that prove we can't have two solutions?

(d) Can you explain informally why the equation can only have one solution, in terms of rates of change?

(e) Bonus: could you make the same argument about $g_2(x) = x^5 + x^3 - 1$? Why or why not? What would change?

Solution:

(a)

$$g(-2) = -43 \qquad g(-1) = -4 \qquad g(0) = -1$$
$$g(1) = 2 \qquad g(2) = 41 \qquad g(3) = 272$$

It looks like this function should have a root between $x = 0$ and $x = 1$, but probably not any others.

(b) We know the function g is continuous on the closed interval $[0, 1]$, and we know that $g(0) = -1$ and $g(1) = 2$. Since $-1 < 0 < 2$, then by the Intermediate Value Theorem, there is a c in $(0, 1)$ with $g(c) = 0$.

(c) We know that g is continuous and differentiable everywhere, which lets us use Rolle's Theorem.

Suppose that g has two roots, $a < b$, so that $g(a) = g(b) = 0$. Then g is continuous and differentiable on $[a, b]$ and $g(a) = g(b)$, so by Rolle's Theorem, there is a c in (a, b) so that $g'(c) = 0$.

But $g'(x) = 5x^4 + 3x^2 + 1 \geq 1$, so we can't have a c with $g'(c) = 0$. So we couldn't have two distinct roots for g.

(d) Informally we can say that $g'(x)$ is always positive, so g must always be increasing. And since the value is always increasing, we can't output zero twice: once we've output zero, every further output must be larger.

(e) Not quite! We could make exactly the same argument about the Intermediate Value Theorem: $g_2(0) = -1$ and $g_2(1) = 1$ so by the intermediate value theorem, $g_2(x) = 0$ must have a solution. But $g_2'(x) = 5x^4 + 3x^2$ can in fact be zero, so the Rolle's Theorem argument doesn't actually work.

(This function does have only one root, but we would need slightly better tools to prove this.)

Problem 3. We can also use the mean value theorem to constrain the possible values for a function. For instance, suppose I have a function f, and all I know is that $f(1) = 10$ and $f'(x) \geq 2$ for every x. I want to know about $f(4)$.

(a) What would this mean in English? How should we think about this physically? What does that tell you about $f(4)$?

(b) Use the Mean Value Theorem to set up an equation relating $f(1), f(4)$, and $f'(x)$. What does it tell you about $f(4)$?

(c) How do those two arguments relate to each other?

(d) is it possible for $f(4) = -30$?

Solution:

(a) Your location at time 1 is 10; maybe this is "your distance from home", so at 1 PM you're ten miles from home. Your speed is always at most 2, meaning that you're going no more that 2 miles per hour away from home. At 4 PM, you'll be at most 16 miles from home.

(b) f is continuous and differentiable, so there's some c such that

$$\begin{aligned}\frac{f(4)-f(1)}{4-1} &= f'(c) < 2 \\ f(4)-10 &< 2\cdot 3 \\ f(4) &< 16.\end{aligned}$$

(c) We did get the same answer both ways! The second argument is just a formalization of the first.

(d) Yes! The not-very-physical idea in this problem is that your speed is always less than 2, but it can be as *negative* as you want. We're essentially allowing infinite speed, but only backwards.

Problem 4. Suppose $|g'(x)| \leq 2$ for all x, and $g(0) = 7$. We want to know about $g(5)$?

(a) What would this mean in English? How should we think about this physically? What does that tell you about $g(5)$?

(b) Use the Mean Value Theorem to set up an equation relating $g(0), g(5)$, and $g'(x)$. What does it tell you about $f(4)$?

(c) How do those two arguments relate to each other?

(d) is it possible for $g(5) = -30$?

Solution:

(a) At time 0 (noon?) we're 7 miles from home. Our speed is never more than 2 miles per hour, so over five hours we'll travel at most ten miles. We could get further from home and be as much as 17 miles away. Or we can go the other direction, overshoot, and be -3 miles away.

(b) g is continuous and differentiable, so there's some c such that

$$\begin{aligned}\frac{g(5)-g(0)}{5-0} &= f'(c) \\ \left|\frac{g(5)-7}{5}\right| &= |f'(c)| < 2 \\ |g(5)-7| &< 10 \\ -10 < g(5)-7 &< 10 \\ -3 < g(5) &< 17.\end{aligned}$$

(c) Again, we got the same answer both ways (if we got the absolute values right).

(d) This time, no. Because we bounded the *absolute value* of our derivative, we couldn't go too fast in either direction.

Problem 5. Let $f(x) = 2x^3 + 3x^2 - 36x$.

(a) Find the critical points of f.

(b) Which of these points can you classify using the second derivative test?

(c) Classify all the critical points using the first derivative test.

Solution:

(a) $f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x + 3)(x - 2)$, so the critical points are -3 and 2.

(b) $f''(x) = 12x + 6$. We see $f''(-3) = -30 < 0$ so f has a local maximum of $f(-3) = 81$ at -3. And $f''(2) = 30 > 0$ so f has a local minimum of $f(2) = -44$ at 2.

(c) We can make a chart

	$6(x+3)$	$x-2$	$f'(x)$
$x < -3$	$-$	$-$	$+$
$-3 < x < 2$	$+$	$-$	$-$
$-3 < x < 2$	$+$	$+$	$+-$

So f is increasing on $(-\infty, -3)$ and $(2, +\infty)$ and is decreasing on $(-3, 2)$. So again we see that f has a local max of $f(-3) = 81$ at -3 and a local min of $f(2) = -44$ at 2.