Math 1231 Spring 2024
Single-Variable Calculus I Section 11
Mastery Quiz 9
Due Tuesday, March 26

This week's mastery quiz has three topics. Everyone should submit all three (sorry).

Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Thursday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Optimization and Extrema
- Secondary Topic 7: Curve Sketching
- Secondary Topic 8: Physical Optimization

Name:

Recitation Section:

Major Topic 3: Optimization and Extrema

(a) The function $f(x) = \dfrac{x^2+5}{x+2}$ has absolute extrema either on the interval $[-3, 0]$ or on the interval $[0, 3]$. Pick one of those intervals, explain why f has extrema on that interval, and find the absolute extrema.

Solution: f is continuous on the closed interval $[0, 3]$, so it must have extrema there. (It is *not* continuous on $[-3, 0]$ because it is undefined at -2.)

We compute

$$\begin{aligned} f'(x) &= \frac{2x(x+2) - (x^2+5)}{(x+2)^2} = \frac{x^2+4x-5}{(x+2)^2} \\ &= \frac{(x+5)(x-1)}{(x+2)^2} \end{aligned}$$

is zero for $x = 1, -5$ and is undefined for $x = -2$, so those are the critical points. The only one we have to care about is $x = 1$.

$$\begin{aligned} f(0) &= 5/2 \\ f(1) &= 2 \\ f(3) &= 14/5 \end{aligned}$$

so the absolute minimum is 2 at 1, and the absolute maximum is 14/5 at 3.

(b) Find and classify the critical points of $f(x) = \sqrt[3]{x^3 - 3x}$.

Solution: We have

$$f'(x) = \frac{1}{3}(x^3-3x)^{-2/3}(3x^2-3) = \frac{(x-1)(x+1)}{\sqrt[3]{x(x^2-3)}^2}.$$

This is zero when $x = \pm 1$ and is undefined when $x = 0$ or $x = \pm\sqrt{3}$. Thus the critical points are $-\sqrt{3}, -1, 0, 1, \sqrt{3}$.

We could try the second derivative test, but we know it won't work at $\pm\sqrt{3}$ or at 0, since those are non-differentiable critical points. So we make a chart:

	$(x-1)$	$(x+1)$	$\sqrt[3]{x^2}$	$\sqrt[3]{(x^2-3)^2}$	f'
$x < -\sqrt{3}$	$-$	$-$	$+$	$+$	$+$
$-\sqrt{3} < x < -1$	$-$	$-$	$+$	$+$	$+$
$-1 < x < 0$	$-$	$+$	$+$	$+$	$-$
$0 < x < 1$	$-$	$+$	$+$	$+$	$-$
$1 < x < \sqrt{3}$	$+$	$+$	$+$	$+$	$+$
$\sqrt{3} < x$	$+$	$+$	$+$	$+$	$+$

So we see the function has a local maximum at -1 and a local minimum at 1; the critical points at $-\sqrt{3}, 0$, and $\sqrt{3}$ are neither minima nor maxima.

Secondary Topic 7: Curve Sketching

Let $g(x) = \dfrac{x^2 - 7}{x^2 - 4}$.

We can compute that $g'(x) = \dfrac{6x}{(x+2)^2(x-2)^2}$ and $g''(x) = \dfrac{-6(3x^2+4)}{(x^2-4)^3}$.

Sketch a graph of the function $g(x)$. Your answer should discuss the domain, asymptotes, roots, limits at infinity, critical points and values, intervals of increase and decrease, points of inflection, and concavity.

Solution: g has domain all reals except ± 2. $g(x) = 0$ when $x = \pm\sqrt{7}$. $\lim_{x\to\pm\infty} \frac{x^2-7}{x^2-4} = \lim_{x\to\pm\infty} \frac{1-7/x^2}{1-4/x^2} = 1$.

The derivative $g'(x) = \frac{6x}{(x+2)^2(x-2)^2}$ is defined except at $x = \pm 2$. It is zero when $x = 0$. The critical points are thus 0 and arguably ± 2; $g(0) = 7/4$.

Since the denominator is a square, it is positive when $x > 0$ and negative when $x < 0$, so it is decreasing for negative x and increasing for positive x. This means that $(0, 7/4)$ is a relative minimum.

We know that

$$g''(x) = \frac{-6(3x^2+4)}{(x^2-4)^3}$$

The numerator is negative everywhere. The denominator is negative when $-2 < x < 2$ and is positive when $x < -2$ or when $2 < x$. Thus the concavity only changes at ± 2; it is concave up on $(-2, 2)$ and concave down elsewhere.

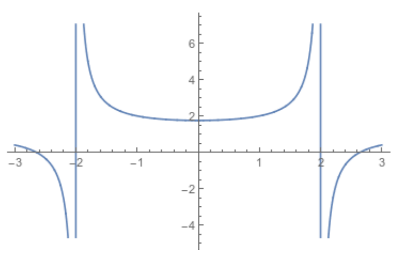

Figure 1: The graph of g from -3 to 3

Secondary Topic 8: Physical Optimization

Suppose that a company that produces and sells x units of a product makes a revenue of $R(x) = 260x - 9x^2/10$ and has costs given by $C(x) = 1000 + 100x + x^2/10$. What is the maximum profit that can be made (where profit is revenues minus costs)?

Solution: Our profit function is $P(x) = R(x) - C(x) = -1000 + 160x - x^2$. Then

$$P'(x)160 - 2x$$
$$160 = 2x$$
$$80 = x$$

We can check that this is truly a maximum by the second derivative: $P''(x) = -2 < 0$ so we have a local maximum.

Or we can see that $P'(x) > 0$ when $x < 80$ and $P'(x) < 0$ when $x > 80$, so the function is increasing until 80 and decreasing after.

The profit at this quantity is

$$P(80) = -1000 + 160(80) - (80)^2 = -1000 + 12800 - 6400 = 5400.$$