§1 Transcendental Functions

§1.1 Invertible Functions

a function is a <u>rule</u> that assigns <u>one</u> <u>output</u> to each <u>input</u> in its domain.

$f(x) = x+3$

$g(x) = x^2+1$

Algebraic

$$\frac{\sqrt[3]{x}+x^5-3}{5x^2+1}$$

Transcendental

sin, cos

logarithm

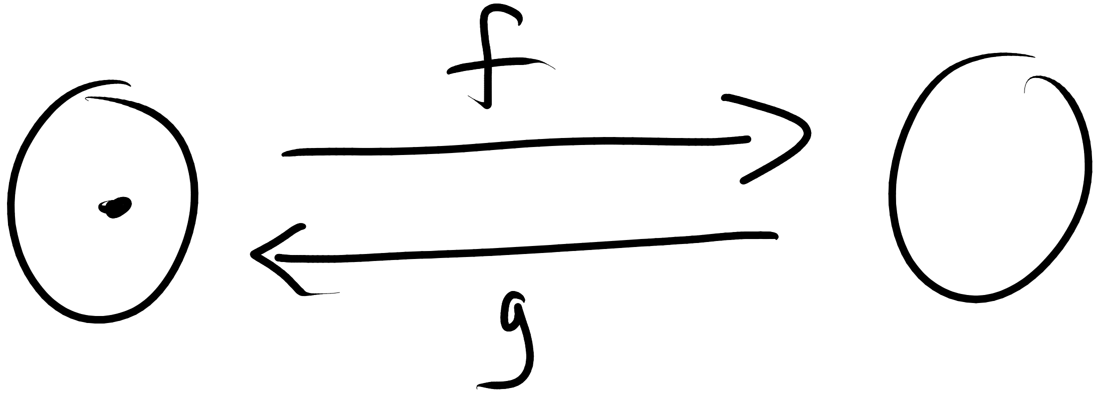
arccos

e^x

$f(x) = x+3$ Q: solve $f(x) = 8$ $x = 5$

$g(x) = x^2+1$ solve $g(x) = 2$ $x = \pm 1.$

Dfn: if f is a fn and $g(f(x)) = x$ for every x in domain of f, then we say g is an <u>inverse</u> of f.



inverse of f is

$$h(x) = x-3.$$

$$h(f(x)) = h(x+3) = (x+3)-3 = x.$$

- If $f(x) = x$

  $g(x) = x.$

- If $f(x) = 5x+3 = y$ $\quad 5x = y-3$

  $g(y) = \dfrac{y-3}{5}$ $\quad x = \dfrac{y-3}{5}$

- $f(x) = x^3$

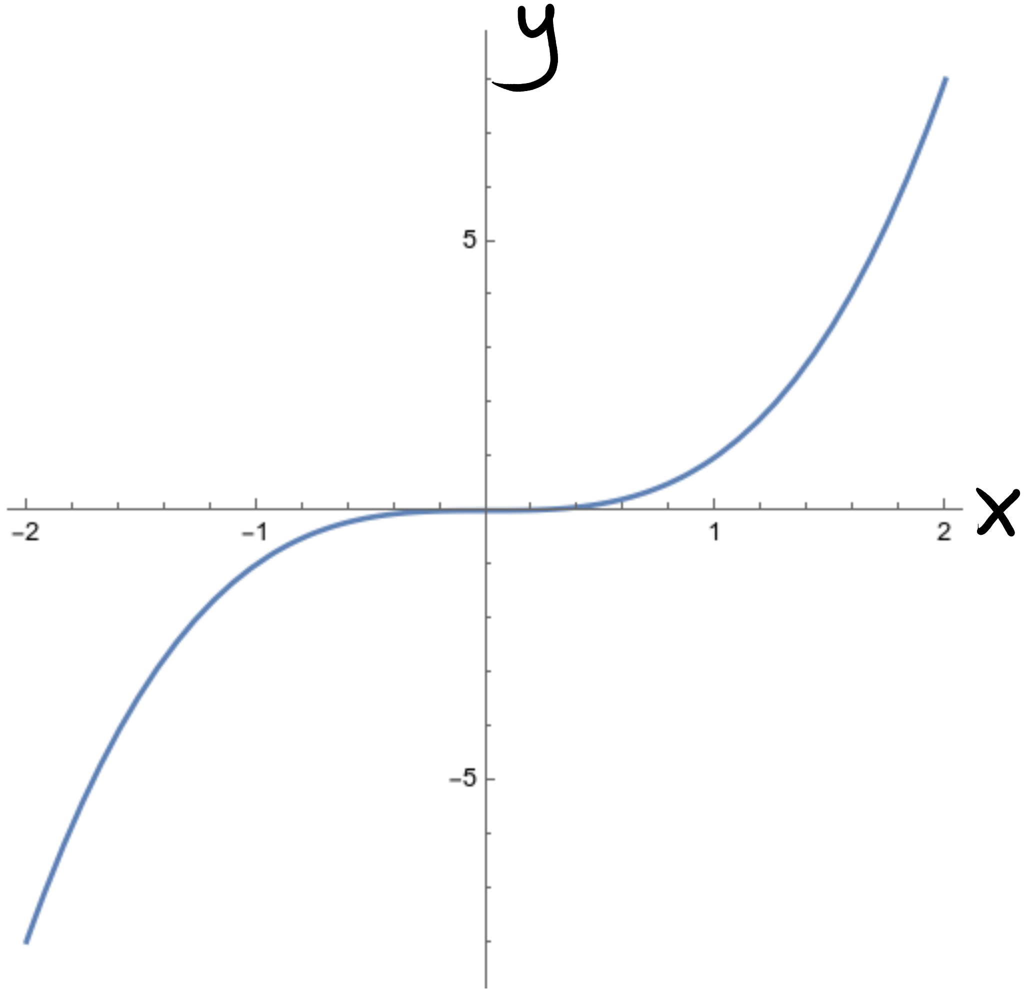
  $g(y) = \sqrt[3]{y}$

$f(x) = x^2+1$

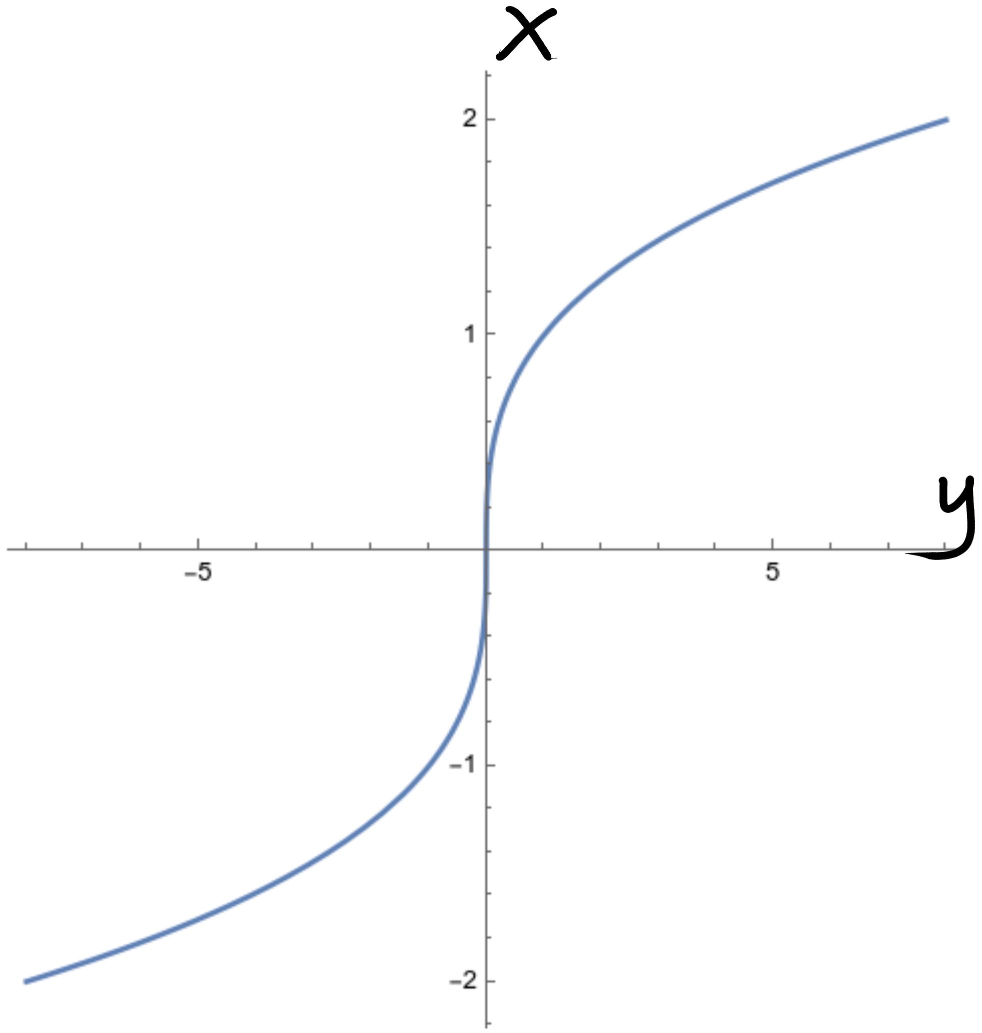
$g(y) = \sqrt{y-1}$

$g(f(-1)) = g(2) = 1.$

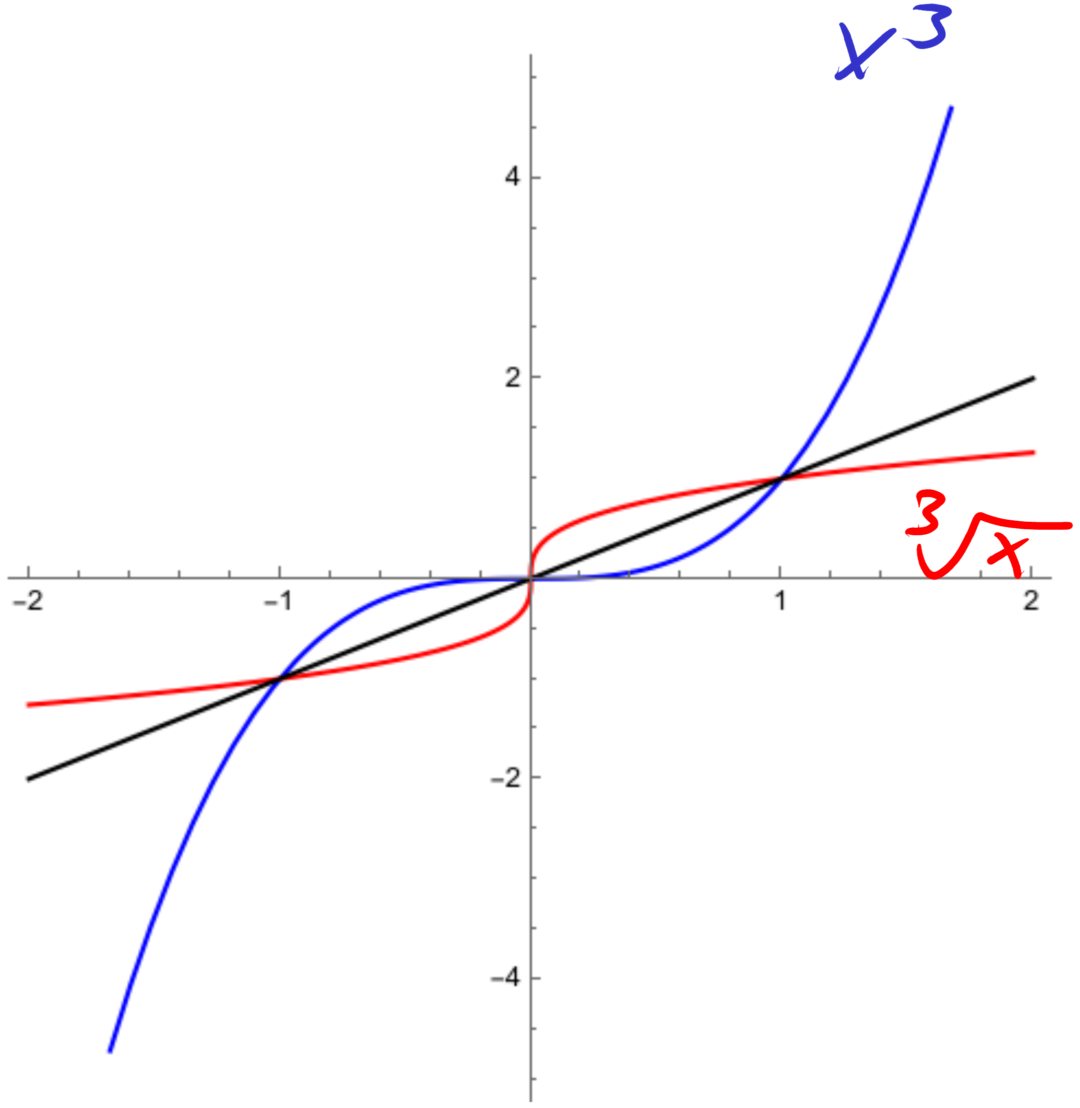
doesn't work



x^3



$\sqrt[3]{x}$



not all fns are invertible

Dfn: a fn is 1-1, one-to-one, injective, ~~monic~~, ~~a monomorphism~~

if whenever $f(a)=f(b)$ we know $a=b$.

no 2 inputs have the same output.

1-1 fns pass the horizontal line test.

$f(x)=x$

If $f(a)=f(b)$ then $a=b$.

$f(x) = 5x+3$

if $f(a) = f(b)$

then $5a+3 = 5b+3$

$5a = 5b$

$a = b.$

so 1-1.

$f(x) = \sqrt{x}$

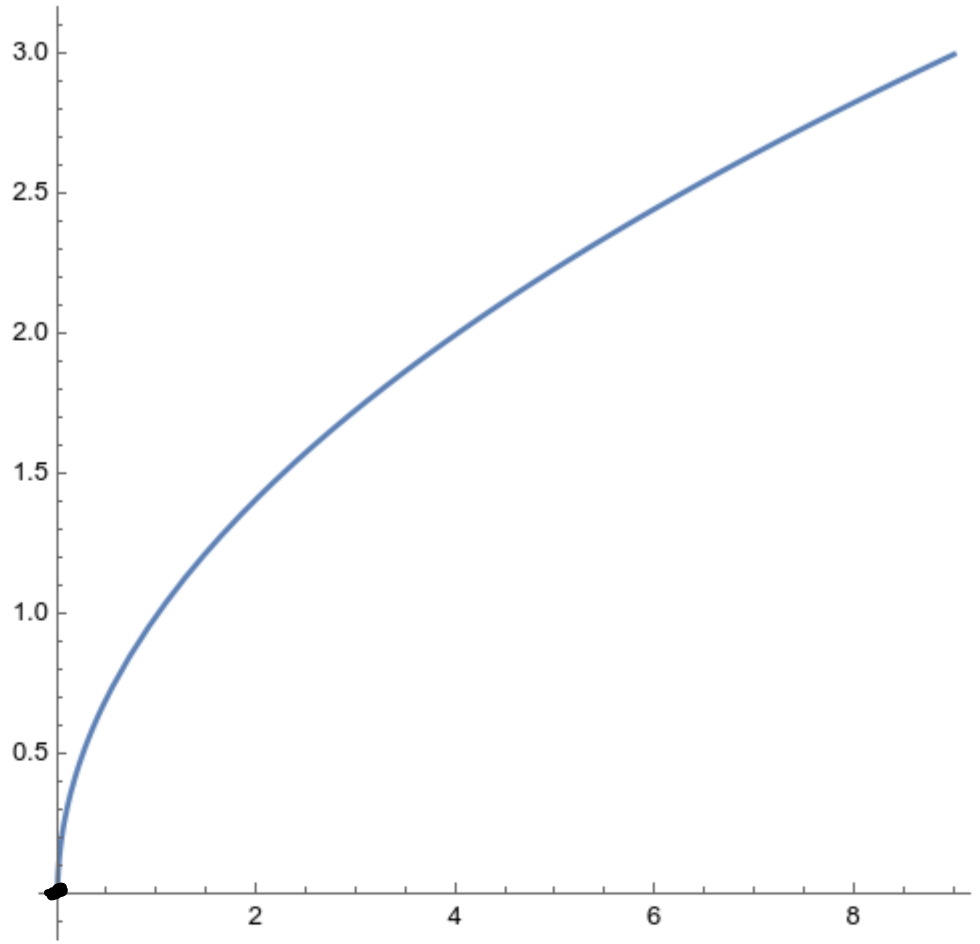
$\sqrt{a} = \sqrt{b}$

then $(\sqrt{a})^2 = (\sqrt{b})^2$

$a = b$

b/c $a, b \geq 0.$

so 1-1.



$f(x) = x^2$

$f(-3) = f(3)$

but $-3 \neq 3$.

not 1-1.

$f(x) = \sin(x)$

$f(0) = f(2\pi) = f(\pi)$

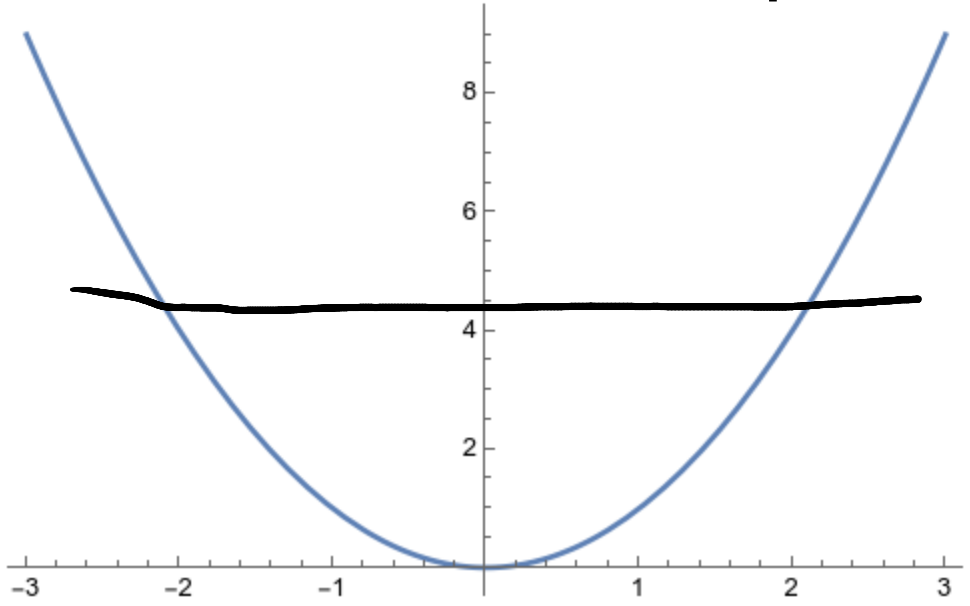
$0 \neq 2\pi \neq \pi$

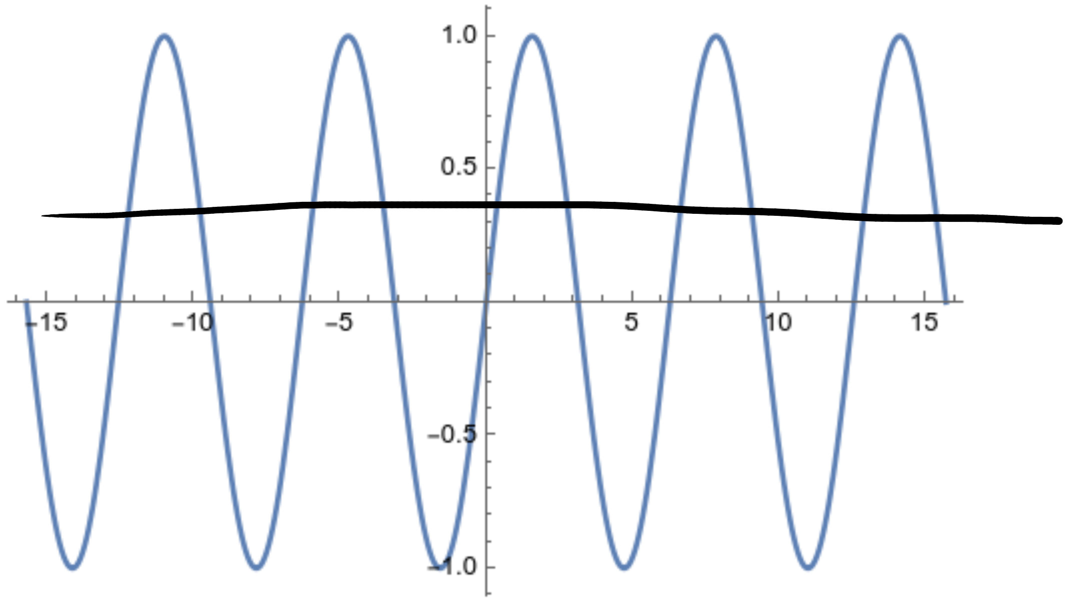
so not 1-1

$f(x) = 3$

$f(17) = f(\pi^3) = 3$.

not 1-1.





not invertible.

If f not 1-1,
f is not invertible.

If f is 1-1,
w/ domain A and image B
there is a fn f^{-1}
w/ domain B and image A
that is the inverse of f.

$$A \underset{f^{-1}}{\overset{f}{\rightleftarrows}} B$$

Restriction of domain

Problem: I want to invert $f(x) = x^2$.

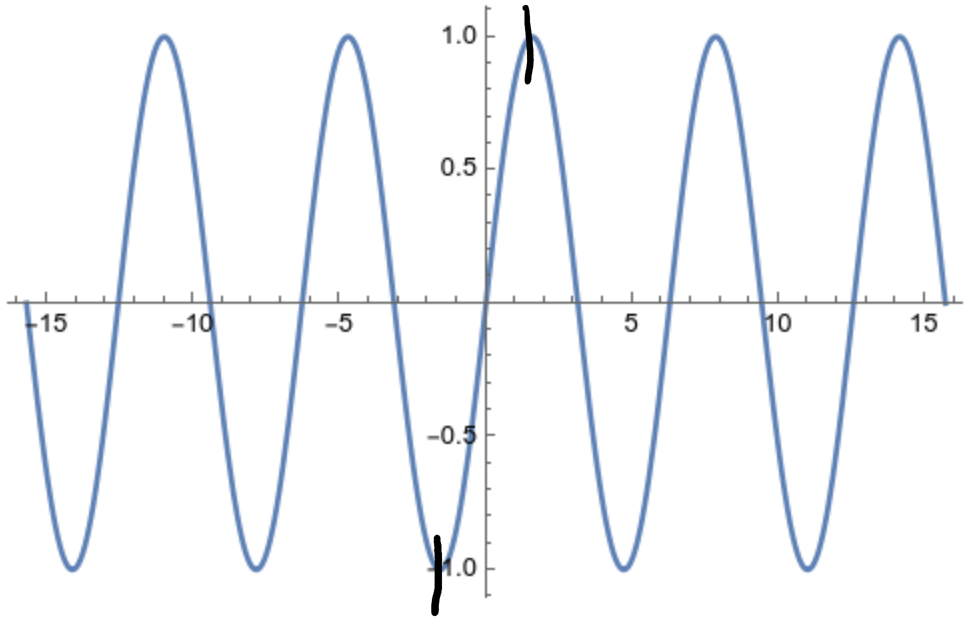
Can we use $\sqrt{x}$?

Problem: only works for $x \geq 0$

$\sqrt{x}$ is an inverse to $f(x)$ for $x \geq 0$.

want to invert $\sin(x)$.

have inverse for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$



§1.1.1 Calc of invertible fns

If f is 1-1 and cts, then f^{-1} is cts.

$$\lim_{x \to a} f(x) = b$$

if x close to a, y close to b

then

$$\lim_{y \to b} f^{-1}(y) = a$$

if y close to b, x is close to a.

Derivatives of inverse fns

$$x \underset{f^{-1}}{\overset{f}{\rightleftarrows}} y \qquad f' = \frac{dy}{dx}$$

$$(f^{-1})' = \frac{dx}{dy} = \frac{1}{dy/dx}$$

Abuse of notation

$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x/\Delta y}$$

Thm (Inverse Function Theorem)

Suppose f is 1-1, differentiable, and $f'(f^{-1}(a)) \neq 0$.

Then $(f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$.

$f(b) = f(f^{-1}(a)) = a$

Pf/ set $y = f^{-1}(x)$, $b = f^{-1}(a)$

$$(f^{-1})'(a) = \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x-a} = \lim_{y \to b} \frac{y-b}{f(y)-f(b)}$$

$$= \lim_{y \to b} \frac{1}{\frac{f(y)-f(b)}{y-b}} = \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))}.$$