1 Transcendental Functions

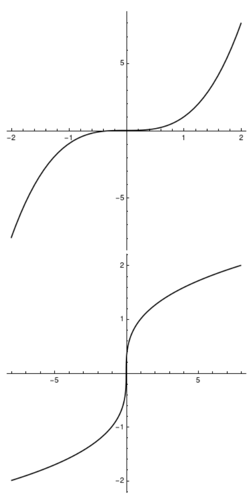
1.1 Invertible functions

Recall that a *function* is a rule that takes an input and assigns a specific output. Sometimes we want to undo this process. This is in fact a natural question; "What do I have to do if I want to get X" is a pretty common thought process. So our goal is: given a function f, given $f(x)$, can we find x?

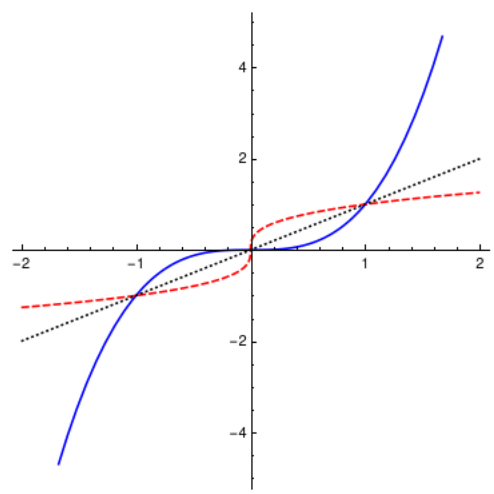
Definition 1.1. If f is a function and $(g \circ f)(x) = x$ for every x in the domain of f, then we say g is an *inverse* of f.

Example 1.2.
- If $f(x) = x$ then $g(y) = y$ is an inverse to f.
- If $f(x) = 5x + 3$ then $g(y) = (y - 3)/5$ is an inverse to f.
- If $f(x) = x^3$ then $g(y) = \sqrt[3]{y}$ is an inverse to f.

Graphically, the graph of f^{-1} looks like the graph of f flipped across the line $y = x$, which makes sense, since a point (x, y) on the graph of f should correspond to a point (y, x) on the graph of f^{-1}.



Top: x^3. Bottom: $\sqrt[3]{x}$. See how they mirror each ohter.



The graph of x^3 in solid blue, and the graph of $\sqrt[3]{x}$ in dashed red. Notice they are mirrored across the dotted black line $y = x$.

Remark 1.3. A given function f has at most one inverse—if f has an inverse at all, then that means "for any y, find the x where $f(x) = y$" is a well-defined rule.

If g is an inverse to f, then the domain of g is the image of f and the domain of f is the image of g.

Computing $f^{-1}(y)$ is the same as solving the equation $f(x) = y$.

Unfortunately, we can't always find these inverses. For instance, if you know that $x^2 = 9$, you don't know for sure what x is: it could be 3 or -3. Similarly, if you know $\sin(x) = 0$, then x could be $n\pi$ for any integer n. The fundamental problem here is that there are some outputs that are generated by more than one input.

Definition 1.4. A function f is *1-1* or *one-to-one* (or *injective*) if, whenever $f(a) = f(b)$, we know that $a = b$.

Example 1.5. Functions which are 1-1:

- $f(x) = x$. If $f(a) = f(b)$ then $a = b$ by definition.
- $f(x) = x^3$. If $f(a) = f(b)$ then $a^3 = b^3$, and then $(a/b)^3 = 1$ so $a/b = 1$ and $a = b$.
- $f(x) = \sqrt{x}$. If $f(a) = f(b)$ then $\sqrt{a} = \sqrt{b}$ so $|a| = |b|$. But $a, b \geq 0$ since they're in the domain of f, and thus $a = b$.

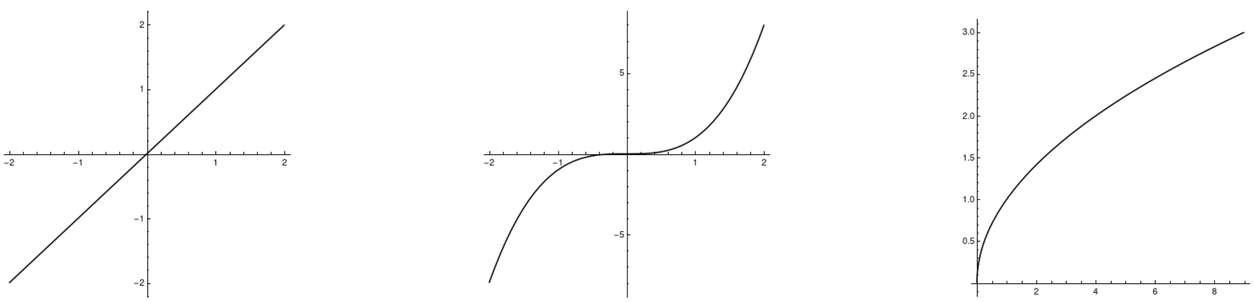


Figure 1.1: Some one-to-one functions: $f(x) = x, f(x) = x^3, f(x) = \sqrt{x}$

Functions which are not 1-1:

- $f(x) = x^2$, since $f(-1) = f(1)$.
- $f(x) = |x|$, since $f(-2) = f(2)$.
- $\sin(x)$, since $\sin(0) = \sin(\pi)$.
- $f(x) = 3$, since $f(a) = f(b) = 3$ for any real numbers a and b.

We might also want to think about what being one-to-one means for the graph of a function. We can't have two inputs with the same output, which means we can't have the same horizontal position at two different points.

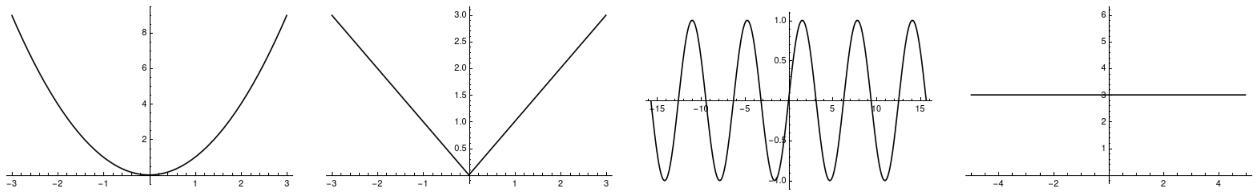


Figure 1.2: Some not one-to-one functions: $f(x) = x^2, f(x) = |x|, f(x) = \sin(x), f(x) = 3$.

Proposition 1.6 (Horizontal Line Test). *A function f is 1-1 if and only if any horizontal line will intersect its graph in at most one point.*

We can see this on the graphs above: all the one-to-one graphs pass the horizontal line test, and all the not one-to-one graphs fail it. We can also interpret it in terms of the reflection property: a function passing the horizontal line test is the same as its reflection/inverse passing the vertical line test.

We already saw that every function with an inverse must be one-to-one, since otherwise there's not a unique answer to the inverse question. Less obvious is that being 1-1 is enough to be invertible, but it's true.

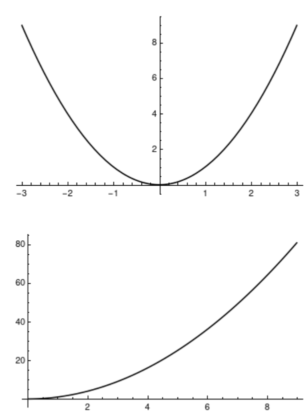
Proposition 1.7. *If f is a 1-1 function with domain A and image B, then there is a function f^{-1} with domain B and image A which is an inverse to f.*

Thus we know now exactly which functions have inverses. However, a lot of functions we would *like* to invert are not one-to-one, which causes a problem. We can often solve this problem by restricting the domain of a function to force it to become one-to-one.

Example 1.8.

We want $\sqrt{x}$ to be the inverse of x^2, but it really isn't. We know that $\sqrt{x^2} = x$ *if* $x \geq 0$, but if x is a negative number this doesn't work. The function $f(x) = x^2$ isn't one-to-one, and thus isn't invertible.

But consider the function $f(x) = x^2$ on the domain $[0, +\infty)$. We can prove this function is one-to-one: if $f(a) = f(b)$ then $a^2 = b^2$ so $a = \pm b$. But both $a, b \geq 0$ so $a = b$. And in fact $\sqrt{x}$ is an inverse to the function $f(x) = x^2$ defined on the domain $[0, +\infty)$.

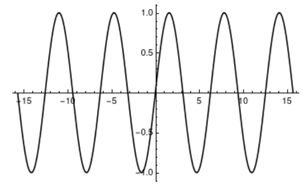


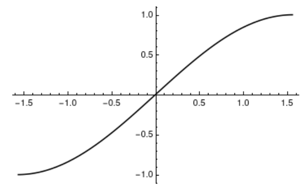
Example 1.9.

We saw that $\sin(x)$ isn't invertible. For instance, $\sin(n\pi) = 0$ for any whole number n.

But if we consider the function $\sin(x)$ restricted to the domain $[-\pi/2, \pi/2]$, it is in fact one-to-one. If we look at the unit circle, we see that as x varies from $-\pi/2$ to $\pi/2$, the y coordinate on the unit circle is always increasing, and so never repeats itself.

Thus we can find an inverse to the sine function on the domain $[-\pi/2, \pi/2]$; we will discuss this further in section 1.5.



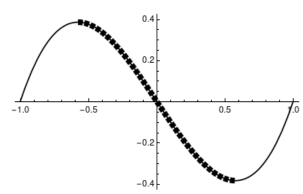


We can find the inverse to a function by writing the equation $y = f(x)$ and solving for x as a function of y. (Sometimes we instead write $x = f(y)$ and solve for y as a function of x; it depends on how we're thinking of the function and what we plan to use it for.) This is also a good way to prove that f is one-to-one.

Example 1.10. Let $f(x) = x^4$ with domain $(-\infty, 0]$. Then we have $y = x^4 \Rightarrow x = \pm\sqrt[4]{y}$. But we know that $x < 0$ so $x = -\sqrt[4]{y}$, and thus $g(y) = -\sqrt[4]{y}$ is an inverse for f.

Example 1.11.

Take $f(x) = x^3 - x$. This function is clearly not one-to-one, since $f(1) = f(0) = f(-1) = 0$. But we can split it up into intervals where it is one-to-one. Looking at the graph, it seems natural to split it up at the critical points. And this suggests we should use calculus to study our inverse function problem.



1.1.1 Calculus of inverse functions

Now that we understand inverse functions as functions, we'd like to see what calculus can tell us about them.

Proposition 1.12. *If f is one-to-one and continuous at a, then f^{-1} is continuous at $f(a)$.*

If f is one-to-one and continuous, then f^{-1} is continuous.

We'd really like to know about the derivatives of inverse functions. We can work out what they are with some quick sketched arguments, and then can prove the answer rigorously once we know what we're looking for.

First, the argument by "it looks nice in the notation": we can rephrase this theorem as saying that

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Second, if we already know that both functions are differentiable, we can use implicit differentiation:

$$\begin{aligned} f^{-1}(f(x)) &= x \\ (f^{-1})'(f(x)) \cdot f'(x) &= 1 \\ (f^{-1})'(f(x)) &= \frac{1}{f'(x)}. \end{aligned}$$

Writing $x = f^{-1}(a)$, or equivalently $a = f(x)$, gives our statement.

Theorem 1.13 (Inverse Function Theorem)**.** *If f is a one-to-one differentiable function, and $f'(f^{-1}(a)) \neq 0$, then $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$.*

Proof. Set $y = f^{-1}(x)$ and $b = f^{-1}(a)$. Then

$$\begin{aligned} (f^{-1})'(a) &= \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a} \\ &= \lim_{y \to b} \frac{y - b}{f(y) - f(b)} \\ &= \lim_{y \to b} \frac{1}{\frac{f(y) - f(b)}{y - b}} \\ &= \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))}. \end{aligned}$$

$\square$

Graphically, this result tells us that the tangent line to f^{-1} at a point has a slope reciprocal to the slope of the tangent line to f at that same point. Really, the tangent line is just being reflected with the graph of the function.

Example 1.14. Let $f(x) = x^n$ on $[0, +\infty)$; then $f^{-1}(x) = \sqrt[n]{x}$. Our formula gives

$$\begin{aligned} (f^{-1})'(a) &= \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(\sqrt[n]{(a)})} \\ &= \frac{1}{n(\sqrt[n]{(a)})^{n-1}} = \frac{1}{na^{(n-1)/n}} = \frac{1}{n} a^{(1-n)/n} = \frac{1}{n} a^{\frac{1}{n} - 1}. \end{aligned}$$

Though at first this didn't look like our original answer, it is the same as the formula we had before.

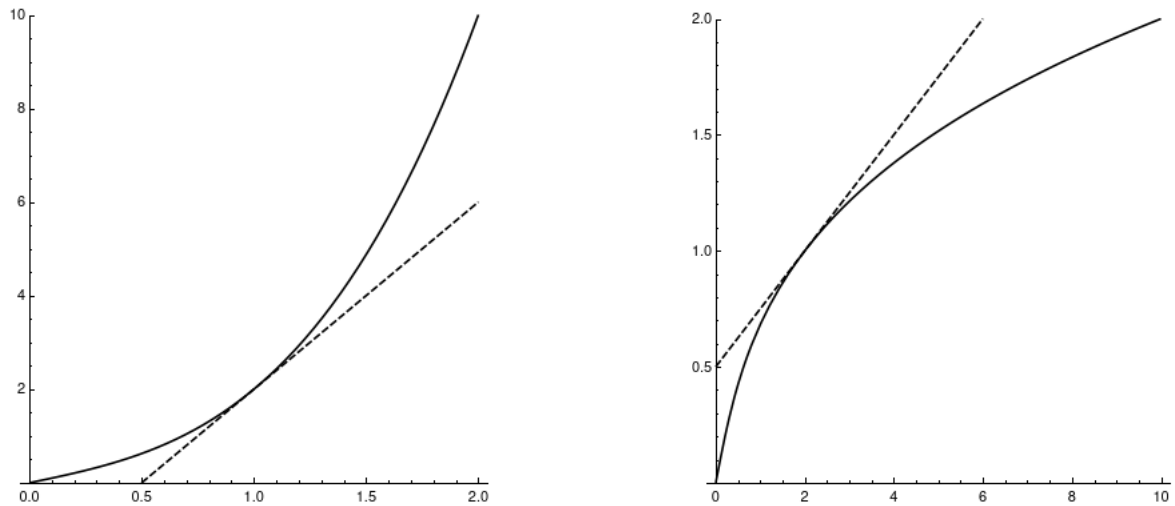


Figure 1.3: Left: the graph of $f(x) = x^3 + x$ with the tangent line at $(x, y) = (1, 2)$. Right: the graph of $f^{-1}(y)$ with the tangent line at $(y, x) = (2, 1)$.

Example 1.15. Let $f(x) = \sqrt[3]{5x^2 + 7}$. What is $(f^{-1})'(3)$?

Well, we have $(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))}$. We know that $f'(x) = \frac{1}{3}(5x^2 + 7)^{-2/3} \cdot 10x$, and we can work out that $f(2) = \sqrt[3]{20 + 7} = 3$ (by plugging in small integers until one works). Thus $f^{-1}(3) = 2$, and so we have

$$(f^{-1})'(3) = \frac{1}{\frac{1}{3}(27)^{-2/3} \cdot 20} = \frac{3 \cdot 9}{20} = \frac{27}{20}.$$

1.2 The exponential and the logarithm

In this section we'll look at a specific, extremely important example: the exponential function $e^x = \exp(x)$ and its inverse the logarithm.

1.2.1 The Exponential

By now we should be familiar with the function $f(x) = x^n$. It's simple to define x^n when n is a positive integer, as $x \cdot x \cdot \dots \cdot x$. It's now clear that we defined $x^{1/n}$ as the inverse function to x^n, with domain restricted to positive numbers in the case n is even and thus x^n is not one-to-one. But can we make sense of x^r where r is any real number? What would it mean to write $2^{\sqrt{2}}$?

The answer would presumably be between 2 and 4. And also between $2^{1.4}$ and $2^{1.5}$. And between $2^{1.41}$ and $2^{1.42}$. In fact, this is how we will define $2^{\sqrt{2}}$. It turns out that there will be exactly one number greater than $2^1, 2^{1.4}, 2^{1.41}, 2^{1.414}, 2^{1.4142}, \dots$ and less than $2^2, 2^{1.5}, 2^{1.42}, 2^{1.415}, 2^{1.4143}, \dots$.

And if this sounds like the approximation-by-zooming in we did with the intermediate value theorem, you're right! If x is a rational or decimal approximation to the real number r, then 2^x should be an approximation to 2^r, and as x gets closer to r the approximation should get better. Thus we get the following definition:

Definition 1.16. If r is any real number, and a is a positive real number, we define $a^r = \lim_{x\to r} a^x$ for x varying over the rational numbers. We say that a is the *base* and r is the *exponent*.

Remark 1.17. We can't actually raise a negative real number to an irrational power. The limit would vary over x with even denominator, and a^x is not defined if x has even denominator and $a < 0$.

Proposition 1.18. *The exponential function $f_a(x) = a^x$ is well-defined for any r when $a > 0$, and is continuous on all real numbers. Further, it satisfies the exponential laws:*

- $a^{x+y} = a^x a^y$
- $a^{x-y} = \frac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$.

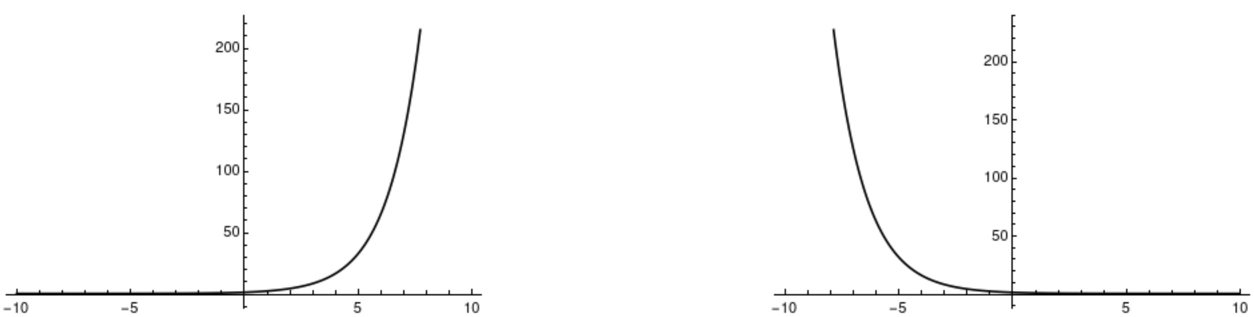


Figure 1.4: The graphs of the exponential functions 2^x and $(1/2)^x$

Proposition 1.19. *If $a > 1$, then $\lim_{x\to+\infty} a^x = +\infty$ and $\lim_{x\to-\infty} a^x = 0$.*

If $0 < a < 1$ then $\lim_{x\to+\infty} a^x = 0$ and $\lim_{x\to-\infty} a^x = +\infty$.

Proof. Both of these can be seen by considering cases where x is an integer. (Or by looking at the graphs.)

When $a > 1$ (say, if $a = 2$, as in figure 1.4), if x is very big then a^x will be very big, and if x is very negative then a^x will be the reciprocal of a very large number, and thus close to 0.

When $0 < a < 1$ (say if $a = 1/2$), if x is very big then a^x will be very close to zero. And if x is very negative then a^x is the reciprocal of a number close to zero, but still positive, and so a^x will be very big. □

There is a number which we will see works much better as a base for the exponential function than any other. This is the number

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

It's possible to prove that this limit exists, but not incredibly easy. It happens that $e \approx 2.71828$. We often write exp for the exponential function with base e; that is, $\exp(x) = e^x$.

Remark 1.20. The number e is also called Euler's number, and was discovered by Jacob Bernoulli in the context of compound interest. (The number was named by Leonhard Euler when he used it for logarithms.)

If your interest rate is r and it's compounded n times a year, then the growth rate per year is $(1 + \frac{r}{n})^n$. If the interest is "compounded continuously," your money grows at a rate equal to the limit of this expression as n goes to $+\infty$—which is e^r.

We'd like to compute the derivative of exp, and also of a^x for a positive real number a. This is a bit difficult to do directly, so instead we're going to cheat.

1.2.2 Logarithms

The exponential function $f(x) = a^x$ is one-to-one, since if $f(x) = f(y)$, then $a^x = a^y$, which means that $a^{x-y} = 1$ and so $x - y = 0$. So a^x must have an inverse function, and we can give it a name.

Definition 1.21. The *logarithmic function with base a*, written $\log_a$, is the inverse function to a^x. It has domain $(0, +\infty)$, and its image is all real numbers.

Thus if $a > 0$, we see that $\log_a(a^x) = x$ for every real x,and $a^{\log_a(x)} = x$ for every $x > 0$.

Remark 1.22. Just as there is a natural base e for the exponential, we also most often use e as the base for a logarithm. In this case we call it the *natural logarithm*, denoted ln.

In high school you probably learned that $\log(x)$ means the base-ten logarithm $\log_{10}(x)$. In high school this is definitely true, and it's sometimes true in fields like chemistry, but in other fields it is not true. (Historically, this was more true, since the base-ten logarithm is useful for doing precise calculations by hand; today we use computers instead.)

In computer science, $\log(x)$ usually refers to a base-two logarithm, since binary is very important. In math, $\log(x)$ usually refers to the natural logarithm. In this course I will try to never write $\log(x)$ without specifying a base.

Example 1.23.

- $\log_3(9) = 2$.
- $\log_2(8) = 3$
- $\log_a(1) = 0$ for any $a > 0$.

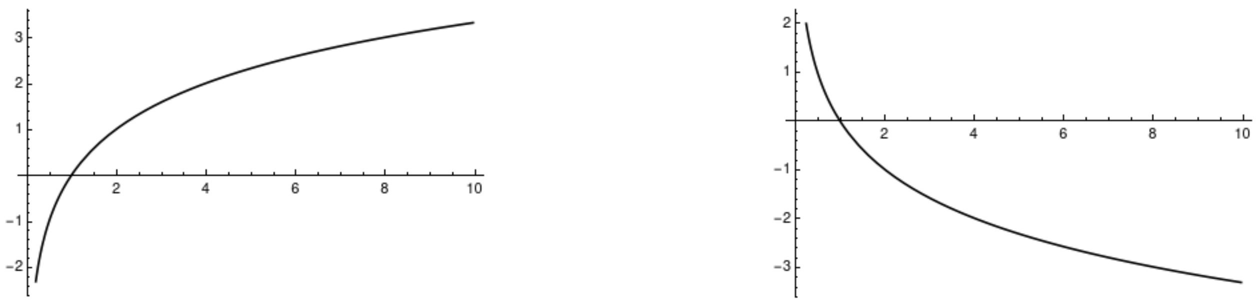


Figure 1.5: The graphs of the exponential functions $\log_2(x)$ and $\log_{1/2}(x)$

Proposition 1.24. *If* $a > 1$, *then* $\lim_{x\to+\infty} \log_a(x) = +\infty$ *and* $\lim_{x\to 0^+} \log_a(x) = -\infty$.

If $0 < a < 1$, *then* $\lim_{x\to+\infty} \log_a(x) = -\infty$ *and* $\lim_{x\to 0^+} \log_a(x) = +\infty$.

Proof. If x is a large number, this means that we're looking for a number y that will make $a^y = x$ large. Looking at the graph of the exponential function, this implies that y must be large if $a > 1$, then y must be very large, and if $0 < a < 1$, then y must be very negative.

If x is very close to 0, we're looking for a y that will maek $a^y = x$ close to 0. If $a > 1$ this happens when y is very negative; if $0 < a < 1$, this happens when y is very positive.

(We can't compute a limit as $x \to -\infty$ since the logarithm is not defined for negative inputs.) $\square$

The logarithm also has a number of properties corresponding to the exponential laws:

Proposition 1.25. *Our exponential laws imply the following logarithm laws:*

- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$

- $\log_a(x^r) = r\log_a(x)$ *for any real number* r.

Proof.
- We can compute that
$$a^{\log_a(x)+\log_a(y)} = a^{\log_a(x)}a^{\log_a(y)} = xy = a^{\log_a(xy)}.$$
Thus the exponents must be the same, and $\log_a(x) + \log_a(y) = \log_a(xy)$.
- We can compute that
$$a^{\log_a(x)-\log_a(y)} = \frac{a^{\log_a(x)}}{a^{\log_a(y)}} = \frac{x}{y} = a^{\log_a(x/y)}.$$
Thus the exponents must be the same, and $\log_a(x) - \log_a(y) = \log_a(x/y)$.
- We can compute that
$$a^{r\log_a(x)} = \left(a^{\log_a(x)}\right)^r = x^r = a^{\log_a(x^r)}.$$
Thus the exponents must be the same, and $r\log_a(x)\log_a(x^r)$.

□

Example 1.26.
- $\ln(a) + \frac{1}{2}\ln(b) = \ln(a) + \ln(b)^{1/2} = \ln(a\sqrt{b})$.
- Solve $e^{5-3s} = 10$. We have that $5 - 3x = \ln 10$ and so $x = \frac{5-\ln 10}{3}$.

Remark 1.27. These properties are actually historically why the logarithm was originally important. Before calculators, people doing difficult computational work had to work by hand. Adding five digit numbers is much, much easier than multiplying them. So engineers would take the log of the numbers, add them together, and then exponentiate. This was all done with the help of massive books called log tables that would tell you the logarithm of a given number. Slide rules are essentially a way of making the log tables portable; but they were superseded by pocket calculators.

There is one more important logarithmic formula, corresponding to the fourth exponential law from proposition 1.18.

Proposition 1.28 (change of base)**.** *For any positive number* $a \neq 1$*, we have* $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$.

Proof. We use the same approach as in proposition 1.25, but now with the natural logarithm. We see that
$$\exp(\log_a(x)\cdot\ln(a)) = (\exp(\ln(a)))^{\log_a(x)} = a^{\log_a(x)} = x,$$
so $\log_a(x)\cdot\ln(a) = \ln(x)$. □

This allows us to convert logs in any base to logs in another base.

Example 1.29. What is $\log_2 10$? By the change of base formula, we have $\log_2(10) = \frac{\ln 10}{\ln 2}$. $\ln 10 \approx 2.3$ and $\ln 2 \approx .7$, so $\log_2 10 \approx 2.3/.7 \approx 23/7$.

1.3 Derivatives of exponentials and logs

Now we're ready to start computing derivatives. The derivative of exp is hard to do directly, so we start with log.

Proposition 1.30. *The function $f(x) = \log_a(x)$ is differentiable, with derivative $f'(x) = \frac{1}{x}\log_a e$.*

Proof.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\log_a(x+h) - \log_a(x)}{h} \\ &= \lim_{h\to 0} \frac{\log_a((x+h)/x))}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\log_a(1 + \frac{h}{x}) \end{aligned}$$

The next step is maybe a little bit of magic, but we want to simplify the inside of the logarithm, so we define a new variable $y = h/x$. This implies that $h = xy$, and we can replace the limit as $h \to 0$ with a limit as $y \to 0$, so we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{1}{h}\log_a(1 + \frac{h}{x}) \\ &= \lim_{\to 0} \frac{1}{xy}\log_a(1+y) \\ &= \frac{1}{x}\lim_{y\to 0} \log_a\left((1+y)^{1/y}\right) \\ &= \frac{1}{x}\log_a\left(\lim_{y\to 0}(1+y)^{1/y}\right). \end{aligned}$$

But note that this entire limit

$$\lim_{y\to 0}(1+y)^{1/y}$$

doesn't depend on x, or on a, which makes it the same for any logarithmic derivative. So we can just give it a name. In fact, we saw it before, at the end of section 1.2.1. This is the number we call e, and is approximately equal to 2.7.

Later on in the course in section 3.3.2 we'll see why this limit is a reasonable one to write down, beyond the fact that it showed up randomly in this calculation. ☐

Corollary 1.31. *If* $f(x) = \log_a(x)$ *then* $f'(x) = \frac{1}{x \ln a}$.

Proof. By the change of base formula, $\log_a(e) = \frac{\ln(e)}{\ln(a)}$. ☐

Corollary 1.32. $\ln'(x) = \frac{1}{x}$.

Example 1.33.

- Let $f(x) = \log_a(x^3 + 1)$. We know that $\log_a'(x) = \frac{1}{x \ln(a)}$. So by the chain rule, we have
$$\frac{d}{dx} \log_a(x^3 + 1) = \log_a'(x^3 + 1) \cdot (x^3 + 1)'$$
$$= \frac{1}{(x^3 + 1)\ln(a)} \cdot 3x^2 = \frac{3x^2}{\ln(a)(x^3 + 1)}.$$
- Let $g(x) = \ln(\cos(x))$. Then $g'(x) = \frac{1}{\cos(x)} \cdot (-\sin(x)) = -\tan(x)$.

  (In section 1.4 we'll see this gives us an antiderivative for $\tan(x)$.)

Example 1.34. The case where $h(x) = \ln|x|$ is very important. If $x > 0$, then $h(x) = \ln(x)$ and so $h'(x) = \frac{1}{x}$. If $x < 0$ then $h(x) = \ln(-x)$, and then $h'(x) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$. So we get the "new" derivative rule:
$$\ln|x| = \frac{1}{x}.$$

This fact will be really important as we start using the logarithm to compute other derivatives and integrals. It allows us to not worry about whether the function we're taking a logarithm of is positive or not; as long as it is non-zero, we can just throw it into $\ln|x|$ and the derivative will come out the same either way.

We can sometimes use logarithms and implicit differentiation to make difficult differentiation problems easier, just as we use them to simplify difficult arithmetic problems.

Example 1.35 (Power Rule)**.** In calculus 1 we stated the power rule $\frac{d}{dx}x^r = rx^{r-1}$, but only really proved it for the case where r is a positive integer (and even that proof was probably fuzzy). In section 1.1.1 we used the inverse function theorem to prove it if $r = 1/q$ for a positive whole number q, but that really only takes us so far.

But with logarithmic differentiation, we can prove the full version without much work. We compute

$$\begin{aligned} y &= x^r \\ \ln|y| &= r\ln|x| \\ \frac{1}{y}\frac{dy}{dx} &= r\frac{1}{x} \\ \frac{dy}{dx} &= r\frac{y}{x} = r\frac{x^r}{x} = rx^{r-1}. \end{aligned}$$

Remark 1.36. When we compute logarithms of two sides of an equation, we often put both sides in absolute values to ensure the logarithm is actually defined. In this example this isn't really necessary since we know x^r must be positive, but in general it's a good safety measure.

And finally, we can use the logarithmic derivatives to figure out the derivative of the exponential, which we couldn't compute directly.

Proposition 1.37. *If $f(x) = a^x$ for $a > 0$, then f is differentiable and $f'(x) = a^x \ln a$.*

Proof.

$$\begin{aligned} y &= a^x \\ \ln|y| &= x\ln|a| \\ \frac{1}{y}\frac{dy}{dx} &= \ln a \\ \frac{dy}{dx} &= y\ln a = a^x \ln a. \end{aligned}$$

□

Corollary 1.38. $\exp'(x) = \exp(x)$. *(Or we can write $\frac{d}{dx}e^x = e^x$.)*

Example 1.39.
- If $f(x) = e^{\sin(x)}$ then $f'(x) = e^{\sin(x)} \cdot \cos(x)$.
- If $g(x) = 5^{x^2+1}$ then $g'(x) = \ln(5)5^{x^2+1} \cdot 2x$.

We see that generally, $\frac{d}{dx}e^y = e^y \cdot y'$.

Remark 1.40. There's another way to think about this process. You could notice that $a^x = e^{x\ln a}$ so

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln(a)} = e^{x\ln(a)} \cdot \ln(a) = a^x \ln(a).$$

This involves the same basic ideas as the logarithmic approach I presented above, but the implementation is slightly different.

If you like this argument or find it more comfortable, go ahead and use it instead of the logarithmic one. But I think the logarithmic version is much easier when the problems get large and complicated.

Example 1.41. If $h(x) = x^x$ we have to be *very careful*—the obvious approaches don't actually work.

There are two ways you could naively try to answer this problem. The power rule, which assumes the exponent is constant, would give $h'(x)" = "x \cdot x^{x-1}$. The recently-learned exponential rule, which assumes the base is constant, would give $h'(x)" = "x^x \cdot \ln(x)$. Neither of these answers is correct, since the exponent and the base are both variable.

But we can solve this logarithmically:

$$\begin{aligned} y &= x^x \\ \ln|y| &= x \ln|x| \\ \frac{1}{y}\frac{dy}{dx} &= \ln|x| + \frac{x}{x} = \ln|x| + 1 \\ \frac{dy}{dx} &= x^x(\ln|x| + 1). \end{aligned}$$

So $h'(x) = (\ln|x| + 1)x^x$.

(If you prefer the exponential approach, you can write $h(x) = e^{x\ln(x)}$, and thus $h'(x) = e^{x\ln(x)}(\ln(x) + 1) = x^x(\ln(x) + 1)$.)

But there is one extra cool thing I want to point out here. If you pretend the base is constant you get $x^x \cdot \ln(x)$. If you assume the exponent is constant, you get $x \cdot x^{x-1}$, which is the same thing as x^x. If you just add these two formulas together—add up the effect of changing the base, and changing the base—then you get

$$x^x \ln(x) + x^x = x^x(\ln(x) + 1)$$

which is indeed the right answer.

We can also use this logarithmic derivative process to simplify derivatives that we could do in other ways.

Example 1.42. We wish to find the derivative of $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

$$\begin{aligned}
\ln y &= \frac{3}{4}\ln(x) + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{3}{4x} + \frac{2x}{2x^2+2} - \frac{3\cdot 5}{3x+2} \\
\frac{dy}{dx} &= y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right) \\
&= \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right).
\end{aligned}$$

1.4 Integrals involving logarithms and exponentials

Computing these derivative formulas also allows us to do some integrals we couldn't do before.

The obvious rule we've gotten is a rule for integrating exponential functions:

$$\int e^x\,dx = e^x + C.$$

Remark 1.43. We could, if we wanted to, treat this as the definition of e^x: it's the unique (up to a constant) function that's its own derivative. It satisfies the *differential equation* $y' = y$. We'll talk more about this idea in section 3.3.

Example 1.44.

- $\int_0^3 e^x\,dx = e^x|_0^3 = e^3 - 1$.
- $\int_0^{\ln(3)} e^x\,dx = e^x|_0^{\ln(3)} = 3 - 1 = 2$.
- Let's compute $\int e^{3x}\,dx$. We can take $u = 3x$ so $dx = du/3$, and we have
$$\int e^{3x}\,dx = \int e^u \frac{du}{3} = \frac{1}{3}e^u + C = \frac{1}{3}e^{3x} + C.$$
- We can approach $\int 3^x\,dx$ in a couple of different ways. One approach is to think about the rule that $\frac{d}{dx}3^x = 3^x\ln(3)$, and thus $\int 3^x\,dx = \frac{3^x}{\ln(3)} + C$.

  The other is to do some algebraic "preprocessing". We know that
$$3^x = (e^{\ln(3)})^x = e^{x\ln(3)}.$$
  Thus we're trying to compute
$$\int e^{x\ln(3)}\,dx = \frac{1}{\ln(3)}e^{x\ln(3)} + C = \frac{1}{\ln(3)}3^x + C.$$

- Let's compute $\int e^x \cos(1+e^x)\,dx$. Here we take $u = 1+e^x$ so $du = e^x\,dx$, and so we want
$$\int \cos(u)\,du = \sin(u) + C = \sin(1+e^x) + C.$$
- Let's compute $\int x^2 e^{x^3}\,dx$. We can take $u = x^3$ so $du = 3x^2 dx$ and we have
$$\int x^2 e^{x^3}\,dx = \int \frac{1}{3} e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C.$$

We learned about the exponential and logarithm, and we learned about the derivative of the exponential and logarithm, so it seems reasonable to think we should now do the integral of the exponential and the logarithm. But that doesn't quite work! The derivative of exp was exp, which allows us to integrate exp. But the derivative of ln was $1/x$; this doesn't actually allow us to integrate ln because we don't have it as the derivative of anything. We will eventually find a way to integrate ln, but that will take tools we don't yet have.

But we do have a much more important integral rule here:
$$\int \frac{1}{x}\,dx = \ln|x| + C.$$
In praactice you can often write $\ln(x)$, but it's safer to write $\ln|x|$; whenever $\ln(x)$ works they mean the same thing, but $\ln|x|$ still works if the denominator of your integrand is negative.

Remark 1.45. Recall that the power rule told us that
$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}$$
whenever $n \neq 1$, but we didn't have a way of integrating x^{-1}. This logarithm rule fills in that gap.

In fact, an alternate path to discover the natural logarithm is to start by trying to find an antiderivative for $\frac{1}{x}$. Some sources will give the definition
$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$
and then define e to be the number that makes this integral equal to 1.

Example 1.46.
- What is the area bounded by $x = 2, x = 3, y = 0$ and $xy = 1$? Drawing the picture, we see we want to compute
$$\int_2^3 \frac{1}{x}\,dx = \ln|x|\Big|_2^3 = \ln(3) - \ln(2) = \ln(3/2) \approx .41.$$

- What is $\int \frac{2x+3}{x^2+3x+5}\,dx$? If we take $u = x^2+3x+5$ then $du = 2x+3dx$, and we have

$$\int \frac{2x+3}{x^2+3x+5}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|x^2+3x+5| + C.$$

- What is $\int \frac{\ln(x)}{x}\,dx$?

  This one looks tricky, but if we take $u = \ln(x)$ so that $du = \frac{1}{x}\,dx$, we see this is

$$\int u\,du = \frac{u^2}{2} + C = \frac{(\ln|x|)^2}{2} + C.$$

Example 1.47. For an even trickier setup, we are finally ready to compute $\int \tan x\,dx$. This isn't obvious at all, but we can see that $\tan x = \frac{\sin x}{\cos x}$; if we take $u = \cos x$, then $du = -\sin(x)\,dx$, and we have

$$\int \tan(x)\,dx = \int -\frac{1}{u}\,du = -\ln|u| + C = -\ln|\cos(x)| + C.$$

Example 1.48 (Recitation challenge)**.** Some integrals here are really truly non-obvious. Suppose we want to compute $\int \frac{dx}{1+e^x}$.

The obvious thing to do is to set $u = e^x$. Then $du = e^x\,dx$ so $dx = du/e^x = du/u$. Then we have

$$\int \frac{dx}{1+e^x} = \int \frac{du/u}{1+u} = \int \frac{du}{u(1+u)}.$$

Using techniques we'll see in a couple weeks, we can work out that $\frac{1}{u(1+u)} = \frac{1}{u} - \frac{1}{u+1}$, and thus the integral is

$$\int \frac{du}{u} - \frac{du}{u+1} = \ln|u| - \ln|u+1| + C = \ln|e^x| - \ln|e^x+1| + C.$$

Alternatively, after some playing around, we can multiply the top and bottom by e^{-x} to get $\int \frac{e^{-x}}{e^{-x}+1}\,dx$. Then we take $u = e^{-x}$ with $du = -e^{-x}\,dx$ so we have

$$\int \frac{e^{-x}}{e^{-x}+1}\,dx = \int \frac{-du}{u+1} = -\ln|u+1| + C = -\ln|e^{-x}+1| + C.$$

It's not at all obvious, but a good exercise, to check that these are the same answer!

1.5 Inverse Trigonometric Functions

We can invert some polynomials, and we can invert exponential functions. The other common category of transcendental functions that we work with is the trigonometric functions, and we'd like to find inverses to these as well.

As a straightforward question, we cannot invert the trigonometric functions because they are all periodic, and thus not one-to-one. For instance, $\sin(0) = \sin(\pi) = \sin(2\pi) = \sin(n\pi)$ for any integer n.

However, sometimes a function is invertible if you restrict its domain enough, to avoid including multiple inputs with the same output. (Often you can achieve this by looking only between two critical points.)

In this section we make canonical domain choices for the trigonometric functions such that they are invertible.

Definition 1.49. If $-1 \leq x \leq 1$, we define:

$$\arcsin(x) = \sin^{-1}(x) = y \text{ where } \sin(y) = x \text{ and } -\pi/2 \leq y \leq \pi/2.$$

The function arcsin has a domain of $[-1, 1]$ and a range of $[-\pi/2, \pi/2]$.

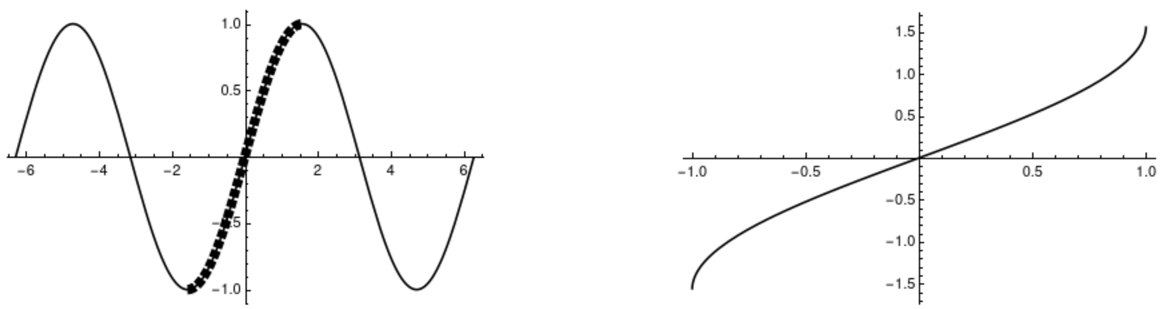


Figure 1.6: Left: A graph of $\sin(x)$ with the restricted domain highlighted
Right: a graph of $\arcsin(y)$

Example 1.50. We can determine that $\arcsin(-\sqrt{3}/2) = -\pi/3$ since $\sin(-\pi/3) = -\sqrt{3}/2$. (Of course, $\sin(5\pi/3) = -\sqrt{3}/2$ as well, but we can ignore this solution because $5\pi/3 > \pi/2$).

With more cleverness, we can calculate $\cos(\arcsin(1/3))$. Suppose $\theta = \arcsin(1/3)$. Then θ is the angle of a triangle with opposite side of lenght 1 and hypotenuse of length 3; using the Pythagorean theorem we determine that the other side has length $\sqrt{8} = 2\sqrt{2}$. Since $\cos(\theta)$ is the length of the adjacent side over the hypotenuse, we have $\cos(\arcsin(1/3)) = 2\sqrt{2}/3$.

We can make similar definitions for inverse cosine and inverse tangent functions. We do have to be careful about the precise domains and images.

Definition 1.51. If $-1 \leq x \leq 1$, we define

$$\arccos(x) = \cos^{-1}(x) = y \text{ where } \cos(y) = x \text{ and } 0 \leq y \leq \pi.$$

This function has domain $[-1, 1]$ and range $[0, \pi]$.

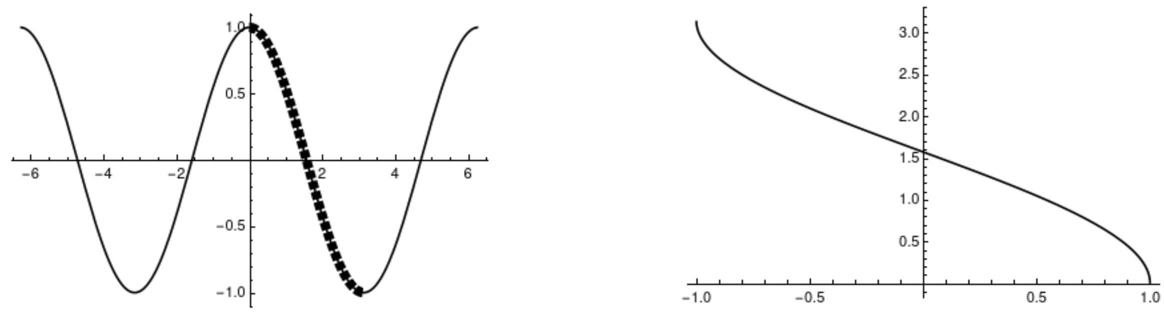


Figure 1.7: Left: A graph of $\cos(x)$ with the restricted domain highlighted
Right: a graph of $\arccos(y)$

Definition 1.52. If x is a real number, we define:

$$\arctan(x) = \tan^{-1}(x) = y \text{ where } \tan(y) = x \text{ and } -\pi/2 < y < \pi/2.$$

This function has domain $(-\infty, +\infty)$ and image $(-\pi/2, \pi/2)$. (Note the strict inequalities $<$ here, rather than the $\leq$ we used for sine and cosine.)

Because the domain here is infinite, we want to think about the limits of this function as well. We know that when x is close to $\pi/2$ then $\tan(x)$ is very large; turning this around, we see that $\lim_{x\to+\infty} \arctan(x) = \pi/2$. similarly $\lim_{x\to-\infty} \arctan(x) = -\pi/2$.

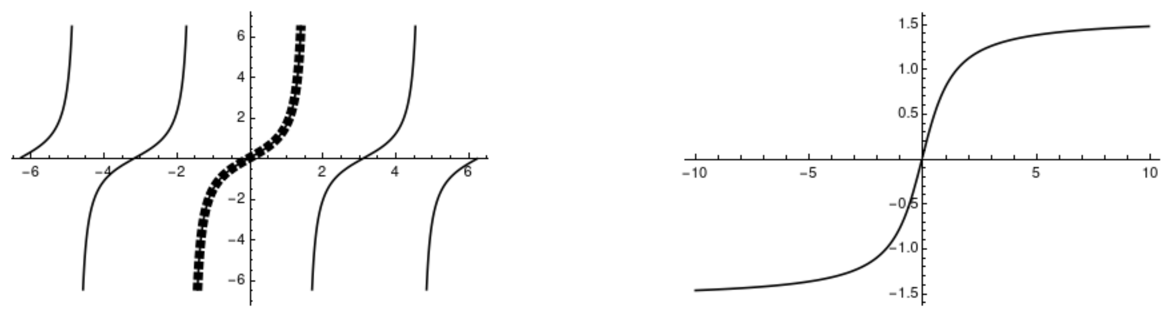


Figure 1.8: Left: A graph of $\tan(x)$ with the restricted domain highlighted
Right: a graph of $\arctan(y)$

The trigonometric functions sin and cos and tan are all differentiable, so by the Inverse Function Theorem 1.13, so are arcsin and arccos and arctan, at least most of the time.

Proposition 1.53. *We have the following derivative formulas:*

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}.$$

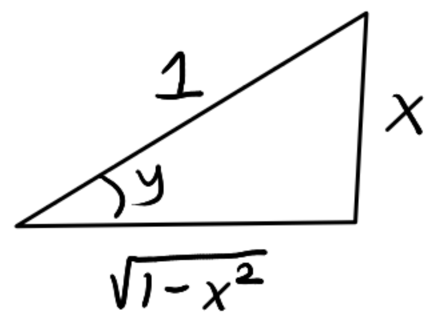
Proof. There are two approaches to proving these facts. One involves trigonometric identities, and the other involves thinking about triangles. They both involve implicit differentiation.

Suppose $y = \arcsin(x)$. Then $\sin(y) = x$ and thus $\cos(y)\frac{dy}{dx} = 1$. Then we have $\frac{dy}{dx} = \frac{1}{\cos(y)}$.

From here, we have two different approaches. One is to note that $\cos(y) = \sqrt{1-\sin^2(y)}$ by the Pythagorean trigonometric identity, and since $y = \arcsin(x)$ we know that $\sin(y) = x$. Thus $\cos(y) = \sqrt{1-x^2}$, and so

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}.$$

I find it easier to think about a different approach, though. If $y = \arcsin(x)$, then y is the angle of a triangle where the opposite side has length x and the hypotenuse has length 1. Then by the Pythagorean theorem, the third side has length $\sqrt{1-x^2}$, so



$$\cos(y) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}.$$

Note we got the same answer both ways, and they both involved basically the same facts; the identity $\sin^2(y) + \cos^2(y) = 1$ holds precisely because of the triangle argument. Either way you want to think of it is fine with me.

We can do the same with $\arccos(x)$. We set $\cos(y) = x$, so

$$\frac{dy}{dx} = \frac{-1}{\sin(y)} = -\frac{1}{\sqrt{1-x^2}}.$$

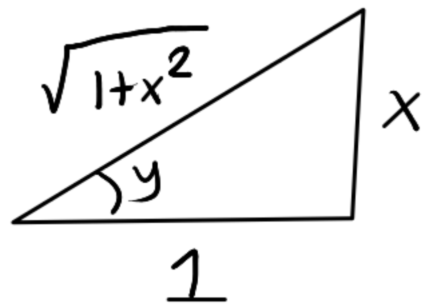
Working out the derivative of arctan is slightly trickier. We set $\tan(y) = x$ so $\sec^2(y)\frac{dy}{dx} = 1$, and thus we have $\frac{dy}{dx} = \cos^2(y)$. We again have two approaches:

First, we can use the identity $1 + \tan^2(y) = \sec^2(y)$, which gives us

$$\cos^2(y) = \frac{1}{\sec^2(y)} = \frac{1}{1+\tan^2(y)} = \frac{1}{1+x^2}$$

since $\tan(y) = x$.

Second, we can see that y is the angle of a triangle with opposite side x and adjacent side 1, and hence hypotenuse $\sqrt{1+x^2}$. Then $\cos(y) = \frac{1}{\sqrt{1+x^2}}$ and so $\arctan'(x) = \cos^2(y) = \frac{1}{1+x^2}$.



$\square$

Example 1.54.

- What is $\arcsin'(3/4)$?

  We know that $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$, so $\arcsin'(3/4)$ is $\frac{1}{\sqrt{1-(3/4)^2}} = \frac{1}{\sqrt{7/16}}$.

- What is $\frac{d}{dx}\arctan(e^x)$?

  Since $\arctan'(x) = \frac{1}{1+x^2}$, we have

$$\frac{d}{dx}\arctan(e^x) = \arctan'(e^x)\cdot(e^x)' = \frac{1}{1+(e^x)^2}\cdot e^x = \frac{e^x}{1+e^{2x}}.$$

- What is $\frac{d}{dx}\arccos(x^2+2x+3)$?

  We get $\dfrac{-1}{\sqrt{1-(x^2+2x+3)^2}}\cdot(2x+2)$.

Remark 1.55. There are also some other derivative formulas that almost no one cares about.

$$\frac{d}{dx}\operatorname{arccot}(x) = \frac{-1}{1+x^2} \qquad \frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\operatorname{arccsc}(x) = \frac{-1}{x\sqrt{x^2-1}}.$$

It's actually a little annoying to define the ranges of these functions, and we mostly avoid using them, but I list the formulas here for completeness.

1.5.1 Integrals with Inverse Trig Functions

These new derivative formulas give us new integral formulas. In practice we only really use two:

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$$
$$\int \frac{dx}{1+x^2} = \arctan(x) + C.$$

The second in particular comes up really often in certain integrals.

Example 1.56. Let's compute $\int \frac{dx}{x^2+4}$.

There are a couple of ways to do this, but the most straightforward is to try to massage it into something with a 1 on the bottom. So we can observe

$$\frac{1}{x^2+4} = \frac{1/4}{x^2/4+1} = \frac{1}{4}\frac{1}{(x/2)^2+1}.$$

So we can take $u = x/2$ with $du = dx/2$, and get

$$\begin{aligned}
\int \frac{dx}{x^2+4} &= \int \frac{1}{4}\frac{1}{(x/2)^2+1}\,dx \\
&= \int \frac{1}{4}\frac{1}{u^2+1}\cdot 2\,du \\
&= \frac{1}{2}\int \frac{1}{u^2+1}\,du \\
&= \frac{1}{2}\arctan(u) + C = \frac{1}{2}\arctan(x/2) + C.
\end{aligned}$$

Example 1.57. Let's compute $\int \frac{x}{\sqrt{1-x^4}}\,dx$.

If the denominator were just $\sqrt{1-x^2}$ this would be a simple u-substitution. But here we need something a bit more.

We can take $u = x^2$ so that $du = 2x\,dx$. Then

$$\begin{aligned}
\int \frac{x}{\sqrt{1-x^4}}\,dx &= \int \frac{x}{\sqrt{1-u^2}}\frac{du}{2x} \\
&= \int \frac{1}{2}\frac{1}{\sqrt{1-u^2}}\,du \\
&= \frac{1}{2}\arcsin(u) + C = \frac{1}{2}\arcsin(x^2) + C.
\end{aligned}$$

Some sources will list the following integral rules for simplicity:

$$\begin{aligned}
\int \frac{dx}{a^2+x^2} &= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \\
\int \frac{dx}{\sqrt{a^2-x^2}} &= \arcsin\left(\frac{x}{|a|}\right) + C.
\end{aligned}$$

There's one more technique we can find useful here, called *completing the square*. This is something almost everyone learned in high school, and then promptly forgot a week later when it was supplanted as a tool for solving equations by the quadratic formula.

Example 1.58. Let's compute $\int \frac{dx}{x^2+2x+5}$.

We want the bottom to look like u^2+1. So first we complete the square to get rid of the $2x$ term. We want to find a number a so that x^2+2x+a is a perfect square; we see that $x^2+2x+1=(x+1)^2$. So we have

$$\begin{aligned}\int \frac{dx}{x^2+2x+5} &= \int \frac{dx}{(x^2+2x+1)+4} = \int \frac{dx}{(x+1)^2+4} \\ &= \frac{1}{4}\frac{dx}{((x+1)/2)^2+1}.\end{aligned}$$

Now we set $u=(x+1)/2$ so $du = dx/2$, and we get

$$\begin{aligned}\int \frac{dx}{x^2+2x+5} &= \int \frac{1}{4}\frac{2du}{u^2+1} = \frac{1}{2}\int \frac{du}{u^2+1} \\ &= \frac{1}{2}\arctan(u)+C = \frac{1}{2}\arctan((x+1)/2)+C.\end{aligned}$$

1.5.2 A note on hyperbolic trigonometric functions

There are some trigonometric-like functions called the hyperbolic trig functions. The basic formulas are

$$\sinh(x) = \frac{e^x-e^{-x}}{2} \qquad\qquad \cosh(x) = \frac{e^x+e^{-x}}{2}$$

and then $\tanh, \coth, \operatorname{sech}, \operatorname{csch}$ are defined as they are for regular trig functions. You can wrok out the derivatives of these functions, and get what you'd maybe expect from the names:

$$\frac{d}{dx}\sinh(x) = \cosh(x) \qquad \frac{d}{dx}\cosh(x) = -\sinh(x) \qquad \frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$$

and so on. We can also define the inverse hyperbolic trigonometric functions and get some familiar-looking formulas, which are occasionally useful:

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}.$$

However, none of these formulas are useful often enough for me to actually want to teach them. It's enough to know that these formulas do exist, and you can look them up if you need to.

As a final note, these definitions don't *look* at all like the regular trig functions, so it's surprising that all the other results work out the same. However, we'll see at the very end of the class that if you allow the imaginary number $i=\sqrt{-1}$, then we can take

$$\sin(x) = \frac{e^{ix}-e^{-ix}}{2i} \qquad\qquad \cos(x) = \frac{e^{ix}+e^{-ix}}{2}.$$

Now we see the relationship between trig and hyperbolic trig functions: we get the hyperbolic trig functions by just ignoring the i terms in these formulas.

1.6 L'Hospital's Rule

We're going to finish by talking about how we can compute limits of transcendental functions like ln and exp. Some of these turn out to be easy:

Example 1.59.

$$\lim_{x\to 1} \frac{\ln(x)^{\nearrow 0}}{x_{\searrow 1}} = \frac{0}{1} = 0.$$

But some of them do not. If we want to compute

$$\lim_{x\to 1} \frac{\ln(x)^{\nearrow 0}}{x-1_{\searrow 0}}$$

we need some more tools.

In general, we only have a problem if our limit is an "indeterminate form", like "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$". There is a very powerful tool we can develop that only works for indeterminate forms; but since indeterminate forms are the only tricky ones, that limitation isn't a real problem.

Theorem 1.60 (L'Hospital's Rule)**.** *Suppose f and g are differentiable, and $g'(x) \neq 0$ near a, except possibly at a. Suppose either $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$. (In other words, the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form). Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right exists.

Remark 1.61. This rule is named after Guillame François Antoine, the Marquis de l'Hospital. (It was discovered by Johann Bernoulli, the brother of the Jacob Bernoulli who discovered e.) Guillame spelled his name "l'Hospital", but later French orthographic reforms shifted the spelling to "l'Hôpital". I follow the more traditional spelling that the Marquis himself used, but you'll see both used interchangeably.

Informal sketch. Let's assume that $f(a) = g(a) = 0$ and $g'(a) \neq 0$. Then by linear approximation we know that $f(x) \approx f(a) + f'(a)(x-a)$, and similarly $g(a) \approx g(a) + g'(a)(x-a)$. Then we have

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)} = \frac{f'(a)(x-a)}{g'(a)(x-a)}$$
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}.$$

□

Proof. Now let's prove this a bit more rigorously, but still staying in the case where $f(a) = g(a) = 0$, $g'(a) \neq 0$, and f' and g' are continuous at a.

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} \\
&= \lim_{x\to a} \frac{(f(x) - f(a))(x-a)}{(g(x) - g(a))(x-a)} \\
&= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} \\
&= \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.
\end{aligned}$$

□

Example 1.62.

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2 - 4x + 3^{\nearrow 0}}{x^2 - 2x - 3_{\searrow 0}} &=^{\text{L'H}} \lim_{x\to 3} \frac{2x-4}{2x-2} = \frac{2}{4} = \frac{1}{2}. \\
\lim_{x\to 0} \frac{1 - \cos(x)^{\nearrow 0}}{\sin(x)_{\searrow 0}} &=^{\text{L'H}} \lim_{x\to 0} \frac{\sin(x)}{\cos(x)} = \frac{0}{1} = 0. \\
\lim_{x\to 1} \frac{\ln x^{\nearrow 0}}{x - 1_{\searrow 0}} &=^{\text{L'H}} \lim_{x\to 0} \frac{1/x}{1} = 1.
\end{aligned}$$

Sometimes we have to apply L'Hôpital's rule more than once to get the results we want.

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan x - x^{\nearrow 0}}{x^3_{\searrow 0}} &=^{\text{L'H}} \lim_{x\to 0} \frac{\sec^2(x) - 1^{\nearrow 0}}{3x^2_{\searrow 0}} \\
&=^{\text{L'H}} \lim_{x\to 0} \frac{2\sec^2(x)\tan(x)}{6x} = \lim_{x\to 0} \sec^2(x) \lim_{x\to 0} \frac{\tan x}{3x} = 1 \cdot \lim_{x\to 0} \frac{\tan(x)^{\nearrow 0}}{3x_{\searrow 0}} \\
&=^{\text{L'H}} \lim_{x\to 0} \frac{\sec^2(x)}{3} = \frac{1}{3}.
\end{aligned}$$

$$\lim_{x\to 0} \frac{e^x - 1 - x^{\nearrow 0}}{x^2_{\searrow 0}} =^{\text{L'H}} \lim_{x\to 0} \frac{e^x - 1^{\nearrow 0}}{2x_{\searrow 0}} =^{\text{L'H}} \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}.$$

We can also use L'Hôpital's rule to evaluate limits at infinity.

Example 1.63.

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{x^2+5x+3^{\nearrow\infty}}{x^2+7x-2_{\searrow\infty}} &=^{\text{L'H}} \lim_{x\to\pm\infty} \frac{2x+5^{\nearrow\infty}}{2x+7_{\searrow\infty}} \\
&=^{\text{L'H}} \lim_{x\to\pm\infty} \frac{2}{2} = 1. \\
\lim_{x\to+\infty} \frac{\ln(x)^{\nearrow\infty}}{x_{\searrow\infty}} &=^{\text{L'H}} \lim_{x\to+\infty} \frac{1/x}{1} = 0. \\
\lim_{x\to+\infty} \frac{e^{x\nearrow\infty}}{x_{\searrow\infty}} &=^{\text{L'H}} \lim_{x\to+\infty} \frac{e^x}{1} = +\infty.
\end{aligned}$$

In fact, it's not too hard to see, using L'Hôpital's Rule, that $\lim_{x\to+\infty} \frac{e^x}{x^n} = +\infty$ and $\lim_{x\to+\infty} \frac{\ln(x)}{x^n} = 0$.

We sometimes say that $\ln(x)$ grows slower than any possible polynomial, and e^x grows faster.

Remember that L'Hôpital's rule only applies if we start with an indeterminate form.

Example 1.64.

$$\begin{aligned}
\lim_{x\to\pi} \frac{\sin(x)^{\nearrow 0}}{1-\cos(x)_{\searrow 2}} &\neq \frac{\cos(x)}{\sin(x)} = \pm\infty \\
\lim_{x\to\pi} \frac{\sin(x)^{\nearrow 0}}{1-\cos(x)_{\searrow 2}} &= \frac{0}{1-(-1)} = 0.
\end{aligned}$$

A more dangerous example:

$$\lim_{x\to 0} \frac{e^x-1-x^{\nearrow 0}}{x^3_{\searrow 0}} =^{\text{L'H}} \lim_{x\to 0} \frac{e^x-1^{\nearrow 0}}{3x^2_{\searrow 0}} =^{\text{L'H}} \lim_{x\to 0} \frac{e^{x\nearrow 0}}{6x_{\searrow 0}}$$

You might think we should use L'Hôpital's rule again here; that would give $\lim_{x\to 0} \frac{e^x}{6} = 1/6$. But the top goes to 1 and the bottom goes to 0, so this is not an indeterminate form! The true limit is $\pm\infty$.

And sometimes L'Hôpital's rule doesn't alwasy work the way we'd like it to, just "because it doesn't."

Example 1.65.

$$\begin{aligned}
\lim_{x\to+\infty} \frac{x^{\nearrow 0}}{\sqrt{x^2+1}_{\searrow 0}} &=^{\text{L'H}} \lim_{x\to+\infty} \frac{1}{\frac{x}{\sqrt{x^2+1}}} = \lim_{x\to+\infty} \frac{\sqrt{x^2+1}^{\nearrow 0}}{x_{\searrow 0}} \\
&=^{\text{L'H}} \lim_{x\to+\infty} \frac{\frac{x}{\sqrt{x^2+1}}}{1} = \lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}}.
\end{aligned}$$

But here if we're clever we can observe that if the limit exists, then

$$\left(\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}}\right)^2 = \lim_{x\to+\infty} \frac{x^2}{x^2+1} = 1$$
$$\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}} = \pm 1.$$

Alternatively, we can just fall back on our techniques from Calculus 1:

$$\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to+\infty} \frac{1}{\sqrt{1+1/x^2}} = \frac{1}{\sqrt{1+0}} = 1.$$

We can often use L'Hôpital's rule to compute limits of other indeterminate forms with a bit of cleverness. Recall the "minor" indeterminate forms are $1^\infty, \infty-\infty, 0^0, \infty^0, 0\cdot\infty$. Products can obviously be rewritten as quotients, and sums or differences can often be combined into something by collecting common denominators. Exponents can be turned into ratios by means of logarithms.

Example 1.66. Our first example is $\lim_{x\to\pi/2} \sec(x) - \tan(x)$, which looks like "$\infty - \infty$". This doesn't require logarithms, but we need to do some pre-processing before we can use L'Hospital's Rule.

$$\begin{aligned}
\lim_{x\to\pi/2} \sec(x) - \tan(x) &= \lim_{x\to\pi/2} \left(\frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)}\right) \\
&= \lim_{x\to\pi/2} \frac{1-\sin(x)^{\nearrow 0}}{\cos(x)_{\searrow 0}} \\
&=^{\text{L'H}} \lim_{x\to\pi/2} \frac{-\cos(x)^{\nearrow 0}}{-\sin(x)_{\searrow 1}} = \frac{0}{1} = 0.
\end{aligned}$$

Example 1.67. The example $\lim_{x\to 0}\cot(2x)\sin(6x)$ looks like "$\infty \cdot 0$". Again, we don't need logarithms, but we do need to do some reorganization before we can use L'Hospital's.

$$\begin{aligned}
\lim_{x\to 0} \cot(2x)\sin(6x) &= \lim_{x\to 0} \frac{\sin(6x)\cos(2x)}{\sin(2x)} = 1 \cdot \lim_{x\to 0} \frac{\sin(6x)^{\nearrow 0}}{\sin(2x)_{\searrow 0}} \\
&= \lim_{x\to 0} \frac{6\cos(6x)^{\nearrow 6}}{2\cos(2x)_{\searrow 2}} = 3.
\end{aligned}$$

Example 1.68. Now let's compute $\lim_{x\to 1} x^{1/(1-x)}$, which looks like "$1^\infty$". Since we have a complicated exponent, this begs for logarithms.

$$\begin{aligned}
y &= x^{1/(1-x)} \\
\ln(y) &= \frac{1}{1-x}\ln(x) = \frac{\ln(x)}{1-x} \\
\lim_{x\to 1}\ln(y) &= \lim_{x\to 1}\frac{\ln(x)^{\nearrow 0}}{1-x_{\searrow 0}} =^{\text{L'H}} \lim_{x\to 1}\frac{1/x}{-1} = -1 \\
\lim_{x\to 1} y &= e^{-1} = \frac{1}{e}.
\end{aligned}$$

Example 1.69. Let's compute $\lim_{x\to+\infty} x^{1/x}$, which looks like "$\infty^0$". Again, we have a complicated exponent, so again we use logarithms.

$$\begin{aligned}
\ln(y) &= \frac{1}{x}\ln(x) = \frac{\ln(x)}{x} \\
\lim_{x\to+\infty}\ln(y) &= \lim_{x\to+\infty}\frac{\ln(x)^{\nearrow\infty}}{x_{\searrow\infty}} =^{\text{L'H}} \lim_{x\to+\infty}\frac{1/x}{1} = 0 \\
\lim_{x\to+\infty} y &= e^0 = 1.
\end{aligned}$$

Example 1.70. Finally, let's compute $\lim_{x\to 0^+} x^{\frac{1}{\ln(x)-1}}$, which looks like "$0^0$".

Again we use logarithms.

$$\begin{aligned}
\ln(y) &= \frac{1}{\ln(x)-1}\ln(x) = \frac{\ln(x)}{\ln(x)-1} \\
\lim_{x\to 0^+}\ln(y) &= \lim_{x\to 0^+}\frac{\ln(x)^{\nearrow\infty}}{\ln(x)-1_{\searrow\infty}} =^{\text{L'H}} \lim_{x\to 0^+}\frac{1/x}{1/x} = 1 \\
\lim_{x\to 0^+} y &= e^1 = e.
\end{aligned}$$

Remark 1.71. The value of 0^0 computed directly is a good question to start bar fights at math conferences. In most non-calculus contexts, the correct answer is 1, but in calculus it depends on exactly which limit you're computing—which is exactly the definition of an indeterminate form.

We can come back to this idea in multivariable calculus: we can say that

$$\lim_{(x,y)\to(0,0)} x^y$$

is indeterminate, and depends on exactly how x and y are related. In example 1.70 we take $y = \frac{1}{\ln(x)-1}$, and then the limit becomes quite determinate—and definitely not equal to 1.

2 Advanced Integration Techniques

In calculus 1 we learned the basics of calculating integrals; in sections 1.4 and 1.5.1 we found some additional formulas that enable us to integrate more functions. But there are still plenty of relatively simple integrals we don't have a way to compute, like

$$\int x\sin(x)\,dx \qquad \int \sin^2(x)\,dx \qquad \int \sqrt{1-x^2}\,dx \qquad \int \frac{1}{x^2-1}\,dx \qquad \int e^{-x^2}\,dx.$$

In this section we'll study some more advanced techniques for finding integrals that will let us handle all of the above questions.

The important skill here isn't simply being able to come up with integral formulas; there are plenty of easy-to-use computer tools that will let you do that. Instead, this material has two goals. First, understanding how some basic, common integrals work makes it easier to intuitively understand applications in other fields. Second and related, I hope learning these techniques illuminates a bit of *why* integrals behave the way they do.

2.1 Integration by Parts

How do we integrate a product of two functions? Sometimes this is easy: if one function is constant, we can just pull it out of the integral; and if one piece is the derivative of the other, we can use u-substitution. But in general we don't have a good way to handle the product of two unrelated functions.

We observed earlier that integrals like addition and scalar multiplication, but don't work well with function multiplication. However, we had a straightforward multiplication rule for derivatives: remember that $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$. We said that every derivative rule gives us an integral rule; can we use this rule somehow to get a product rule for integrals?

Well, what happens if we integrate both sides? It follows from the Fundamental Theorem of Calculus that $\int \frac{d}{dx}f(x)g(x), dx = f(x)g(x) + C$. Thus

$$\int f'(x)g(x) + f(x)g'(x)\,dx = f(x)g(x)$$
$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

Thus if we have an integral we can write as $\int f(x)g'(x)\,dx$, and and we know $\int f'(x)g(x)\,dx$, we can find an antiderivative. This is also sometimes written for bookkeeping as

$$\int u\,dv = uv - \int v\,du.$$

Remark 2.1. I'll mostly compute antiderivatives rather than definite integrals, since you can definitely compute a definite integral if you can find the antiderivative. But we *can* state our result in terms of definite integrals as well:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)\,dx.$$

Example 2.2. Find $\int xe^x\,dx$. We want to write this as a product of two functions, one of which we can easily differentiate and the other we can easily integrate. So let $u = x$ and $dv = e^x\,dx$; then we have $du = 1\,dx$ and $v = e^x$. Thus by integration by parts, we have

$$\begin{aligned} \int u\,dv &= uv - \int v\,du \\ \int xe^x\,dx &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C = (x-1)e^x + C. \end{aligned}$$

Indeed, we can check that $((x-1)e^x)' = e^x + (x-1)e^x = xe^x$.

Example 2.3. Let's find $\int_0^\pi x\cos(x)\,dx$. Again, we note that x becomes simpler when we take a derivative, so we set $u = x, dv = \cos(x)$, and then $du = dx$ and $v = \sin(x)\,dx$. We get

$$\begin{aligned} \int_0^\pi x\cos(x)\,dx &= x\sin(x)\Big|_0^\pi - \int_0^\pi \sin(x)\,dx \\ &= \pi\sin(\pi) - 0\sin(0) - \left(-\cos(x)\Big|_0^\pi\right) \\ &= \cos(\pi) - \cos(0) = -2. \end{aligned}$$

In addition to the "obvious" applications where we're integrating a product, we can sometimes use this method when we have a function with no obvious integral, but whose derivative is much simpler.

Example 2.4 (antiderivative of $\ln(x)$)**.** Consider $\int \ln(x)\,dx$. This doesn't look like integration by parts should help, since it's not a product. But $\ln(x)$ gets much easier to deal with after we take a derivative, so we can try it out. Let $u = \ln(x), dv = 1\,dx$, and thus $du = \frac{1}{x}\,dx$ and $v = x$. We have

$$\begin{aligned} \int u\,dv &= uv - \int v\,du \\ \int \ln(x)\,dx &= x\ln(x) - \int x\frac{1}{x}\,dx \\ &= x\ln(x) - \int 1\,dx = x\ln(x) - x + C. \end{aligned}$$

Indeed, we can check that $(x\ln(x) - x)' = \ln(x) + \frac{x}{x} - x = \ln(x)$.

Integration by parts takes our original integral and replaces it with a new integral we have to do. Sometimes that new integral is also an integral of a product, so we have to repeat the integration by parts process.

Example 2.5. Consider $\int x^2 \cos(x)\, dx$. We can take $u = x^2, dv = \cos(x)\, dx$, so $du = 2x\, dx, v = \sin(x)$. Then

$$\int x^2 \cos(x)\, dx = x^2 \sin(x) - 2 \int x \sin(x)\, dx.$$

We don't really know $\int x \sin(x)\, dx$ either, but we can take $u = x, dv = \sin(x)\, dx$ so $du = dx, v = -\cos(x)$. Then

$$\begin{aligned} \int x \sin(x)\, dx &= -x\cos(x) - \int -\cos(x)\, dx \\ &= -x\cos(x) + \sin(x) + C \\ \int x^2 \cos(x)\, dx &= x^2 \sin(x) - 2\left(-x\cos(x) + \sin(x) + C\right) \\ &= x^2 \sin(x) + 2x\cos(x) - 2\sin(x) + C. \end{aligned}$$

We check our work by taking a derivative, and get

$$2x\sin(x) + x^2\cos(x) + 2\cos(x) - 2x\sin(x) - 2\cos(x) = x^2\cos(x).$$

Example 2.6. Sometimes repeating the integration-by-parts process leads to a repeating cycle; surprisingly, this can still give us an answer.

Consider $\int \sin(x)e^x\, dx$. This is clearly a product, and neither of these becomes particularly simpler or more complex by integrating or differentiating. Still, let's give it a try.

$$\int \sin(x)e^x\, dx = \sin(x)e^x - \int \cos(x)e^x\, dx$$

This doesn't seem to help, because the new integral isn't any easier than the old. Let's keep going anyway.

$$\int \cos(x)e^x\, dx = \cos(x)e^x - \int (-\sin(x)e^x)\, dx$$

and this last integral is the same as the one we started with. This doesn't look promising, but it actually works out fine.

$$\begin{aligned}
\int \sin(x)e^x\,dx &= \sin(x)e^x - \left(\cos(x)e^x + \int \sin(x)e^x\,dx\right) \\
&= \sin(x)e^x - \cos(x)e^x - \int \sin(x)e^x\,dx \\
2\int \sin(x)e^x\,dx &= \sin(x)e^x - \cos(x)e^x + C \\
\int \sin(x)e^x &= \frac{e^x}{2}(\sin(x) - \cos(x)) + C.
\end{aligned}$$

Example 2.7. Consider $\int \sin(x)\cos(x)\,dx$. This is a product, so we can use integration by parts.

$$\begin{aligned}
\int \sin(x)\cos(x)\,dx &= \sin^2(x) - \int \sin(x)\cos(x)\,dx \\
2\int \sin(x)\cos(x)\,dx &= \sin^2(x) + C \\
\int \sin(x)\cos(x)\,dx &= \frac{1}{2}\sin^2(x) + C.
\end{aligned}$$

However, here we could have done something much easier: if $u = \sin(x)$ then $du = \cos(x)\,dx$ and we get

$$\int \sin(x)\cos(x)\,dx = \int u\,du = u^2/2 + C = \frac{1}{2}\sin^2(x) + C.$$

Example 2.8. Finally, consider $\int \cos^2(x)\,dx$. There's no useful u-substitution, but we can look at the integrand as $\cos(x)\cdot\cos(x)$ and use integration by parts. We get

$$\begin{aligned}
\int \cos(x)\cdot\cos(x)\,dx &= \sin(x)\cos(x) - \int -\sin(x)\cdot\sin(x)\,dx \\
&= \sin(x)\cos(x) + \int \sin(x)\cdot\sin(x)\,dx \\
&= \sin(x)\cos(x) + \left(-\sin(x)\cos(x) + \int \cos(x)\cdot\cos(x)\,dx\right) \\
&= 0 + \int \cos(x)\cdot\cos(x)\,dx.
\end{aligned}$$

This is clearly *true* but not at all useful to us. We need to develop some other tools if we want to compute this successfully. So in the next section we'll study how to do trigonometric integrals that aren't already on our formula lists.

2.2 Trigonometric Integrals

2.2.1 Integrals of Trigonometric Functions

So far we've found antiderivatives for a number of trigonometric functions, including sin, cos, tan. Here we study some trigonometric identities that allow us to integrate more difficult functions.

I often say that there are really only two or three trigonometric identities you need to know.

- $\sin^2(x) + \cos^2(x) = 1$.
- $\sin^2(x) = \dfrac{1 - \cos 2x}{2}$.
- $\cos^2(x) = \dfrac{1 + \cos 2x}{2}$.

The second and third are called the "double angle" formulas. I'll call the first the "circle identity" or "pythagorean identity" but that's not a standard name.

Remark 2.9. I say these are the only identities you need to know. However, there are many other identities you can derive from these. As a warning, many problems involving trigonometric functions have multiple solutions which all appear to be different, but are actually the same.

Corollary 2.10. *(a)* $1 + \tan^2(x) = \sec^2(x)$.

(b) $1 + \cot^2(x) = \csc^2(x)$.

We will use these identities to massage our integrals into something doable. For integrals involving powers of sin and cos, our general strategy is to write our integrand as a sum of things with either sin or cos being a first power, and then substituting u for the function with a higher power.

General rule: if you have an odd number of sines or of cosines, you can use the circle identity to get just one sine or just one cosine, which will be your du. If you have an even number of both, use the double-angle formula on all of them to cut the total number in half, until you have an odd number of at least one, then use the circle identity as before.

Example 2.11. If we integrate an even power of sin or cos, we must use the double angle formulas to reduce it.

$$\begin{aligned}
\int \cos^2(x)\,dx &= \frac{1}{2}\int 1+\cos(2x)\,dx \\
&= \frac{1}{2}\int (1+\cos(u))\frac{du}{2} \\
&= \frac{1}{4}(u+\sin(u)) = \frac{x}{2}+\frac{\sin(2x)}{4}+C
\end{aligned}$$

(where $u = 2x, du = 2dx$).

Example 2.12. If we have an odd power of sin or cos, we can use the circle identity to reduce it.

$$\begin{aligned}
\int \sin^3(x)\,dx &= \int \sin(x)(1-\cos^2(x))\,dx \\
&= \int \sin(x) - \sin(x)\cos^2(x)\,dx = \int \sin(x)\,dx - \int \sin(x)\cos^2(x)\,dx \\
&= -\cos(x) + \int u^2\,du = \frac{1}{3}u^3 - \cos(x) = \frac{1}{3}\cos^3(x) - \cos(x) + C
\end{aligned}$$

(where $u = \cos(x), du = -\sin(x)dx$).

Example 2.13.

$$\begin{aligned}
\int \cos^8(x)\sin^3(x)\,dx &= \int \cos^8(x)(1-\cos^2(x))\sin(x)\,dx \\
&= \int \cos^8(x)\sin(x) - \cos^{10}(x)\sin(x)\,dx \\
&= \int u^{10} - u^8\,du = \frac{u^{11}}{11} - \frac{u^9}{9} + C = \frac{\cos^{11}(x)}{11} - \frac{\cos^9(x)}{9} + C
\end{aligned}$$

where $u = \cos(x), du = -\sin(x)\,dx$.

Example 2.14. The trickiest case is when you have even powers of both sine and cosine. Mostly this just leads to a large pile of unpleasant algebra.

$$\begin{aligned}
\int \sin^2(x)\cos^2(x)\,dx &= \int \frac{1}{2}\left(1-\cos 2x\right)\frac{1}{2}\left(1+\cos(2x)\right)\,dx \\
&= \frac{1}{4}\int 1-\cos^2 2x\,dx \\
&= \frac{1}{4}\int 1 - \frac{1}{2}(1+\cos 4x)\,dx \\
&= \frac{1}{4}\left(x - \frac{1}{2}(x+\frac{\sin 4x}{4})\right) \\
&= \frac{3x}{8} - \frac{\sin(4x)}{8} + C.
\end{aligned}$$

Integrals with secant and tangent work slightly differently. We try to use the fact that $1 + \tan^2(x) = \sec^2(x)$ to rewrite the expression so that either there's exactly one tangent, or exactly two secants, and then a u substitution will work.

Example 2.15. If we have an even number of secants we use our identity to rewrite so we have exactly two secants. We will wind up setting $u = \tan\theta$ and $du = \sec^2\theta\, d\theta$.

$$\begin{aligned}
\int \tan^4\theta \sec^4\theta\, d\theta &= \int \tan^4\theta \sec^2(\theta)(\tan^2\theta + 1)\, d\theta \\
&= \int \tan^6\theta \sec^2\theta + \tan^4\theta\sec^2\theta\, d\theta \\
&= \int u^6 - u^4\, du = \frac{u^7}{7} + \frac{u^5}{5} \\
&= \frac{\tan^7\theta}{7} + \frac{\tan^5\theta}{5} + C.
\end{aligned}$$

Example 2.16. If we have an odd number of tangents we use our identity to rewrite things so we have exactly one tangent. We'll set $u = \sec(\theta)$ and $du = \sec(\theta)\tan(\theta)\, d\theta$.

$$\begin{aligned}
\int \tan^3\theta \sec^5\theta\, d\theta &= \int \tan\theta \sec^5\theta(\sec^2\theta - 1)\, d\theta \\
&= \int \tan\theta\sec^7\theta - \tan\theta\sec^5\theta\, d\theta \\
&= \int u^6 - u^4\, du = \frac{u^7}{7} - \frac{u^5}{5} \\
&= \frac{\sec^7\theta}{7} - \frac{\sec^5\theta}{5} + C.
\end{aligned}$$

Remark 2.17. If we have an even number of tangents and an odd number of secants, our life is hard; we usually use integration by parts. For instance,

$$\int \tan^2\theta\sec\theta\, d\theta = \frac{1}{2}\left(\tan\theta\sec\theta + \ln\left(\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right) - \ln\left(\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right)\right).$$

There are a couple of special cases, as well. We observed earlier (by setting $u = \cos\theta, du = -\sin\theta$) that

$$\int \tan\theta\, d\theta = -\ln|\cos\theta| = \ln|\sec\theta| + C.$$

More difficult is the integral of $\sec\theta$. This was a major open problem for most of a century, since it's important in the design of maps and in navigation. It was solved by James Gregroy in 1668, but Isaac Barrow came up with a clearer argument in 1670, by inventing the technique of *partial fractions* which we'll discuss in section 2.3.

For right now I'll just give you the formula:

$$\begin{aligned}
\int \sec\theta \, d\theta &= \ln|\sec\theta + \tan\theta| + C && \text{Gregory's formula} \\
&= \ln\left|\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\right| + C && \text{Numeric conjecture by Henry Bond} \\
&= \frac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| + C && \text{Barrow's formula.}
\end{aligned}$$

All three formulas here are valid. And all three formulas are obnoxious and seem really random.

2.2.2 Trigonometric Substitution

Now that we know how to integrate trigonometric functions, we can often use them to make our lives easier in integrals that don't appear to use trigonometry at all.

Example 2.18. We've known since grade school that the area of a circle with radius r is πr^2. Can we prove this? Consider the function $f(x) = \sqrt{r^2 - x^2}$; this is the graph of a semicircle over the x axis. So we wish to compute $\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$.

There's no way to use integration by parts. We might try setting $u = r^2 - x^2$ but then $du = -2x\, dx$ and we have no way to get rid of the x.

Instead, we do something clever. We notice that $\sqrt{r^2 - x^2}$ looks kind of like $\sqrt{r^2 - (r\sin(x))^2}$, and by the pythagorean identity that would simplify to $\sqrt{r\cos^2(x)} = \cos(x)$. Thus we write $x = r\sin\theta$, and $dx = r\cos\theta\, d\theta$. We get

$$\begin{aligned}
\int \sqrt{r^2 - x^2}\, dx &= \int \sqrt{r^2(1 - \sin^2\theta)} \cdot r\cos\theta\, d\theta \\
&= \int r^2\sqrt{\cos^2\theta} \cdot \cos\theta\, d\theta \\
&= r^2 \int \cos^2\theta\, d\theta \\
&= r^2 \int \frac{1+\cos(2\theta)}{2}\, d\theta \\
&= r^2\left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right).
\end{aligned}$$

At this point we have two choices. One is to change the θ back into x by the formula $\theta = \arcsin(x/r)$. If we do this we find the antiderivative is

$$r^2\left(\frac{1}{2}\arcsin\left(\frac{x}{r}\right) + \frac{1}{4}\sin\left(2\arcsin\left(\frac{x}{r}\right)\right)\right).$$

In principle we can use the double-angle formulas to calculate $\sin(2\arcsin(x/r))$, but in practice this is a huge pain.

But when we do a u-substitution for a definite integral, we can *either* subtitute x back in for u, *or* change the bounds of the definite integral to be in terms of u. The second approach is much easier here. Our original integral was from $-r$ to r; we see that if $x = -r$ then $\theta = -\pi/2$, and if $x = r$ then $\theta = \pi/2$. So we evaluate this integral at $\pi/2$ and $-\pi/2$, and we get

$$\begin{aligned}\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx &= r^2 \left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right)\Bigg|_{-\pi/2}^{\pi/2} \\ &= r^2 \left(\frac{\pi}{4} + \frac{\sin(\pi)}{4} - \left(\frac{-\pi}{4} + \frac{\sin(-\pi)}{4}\right)\right) = r^2 \cdot \frac{\pi}{2}.\end{aligned}$$

Thus the area of the semicircle is $\pi r^2/2$, and so the area of the circle is πr^2.

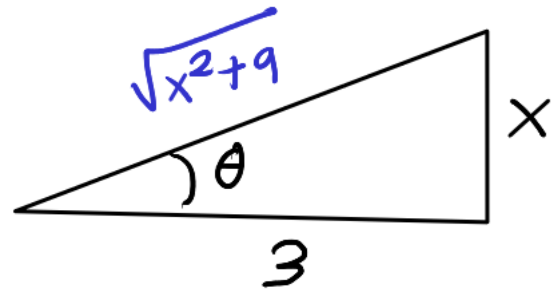
Remark 2.19. In general, this helps when we have a difference of squares under a square root.

- If we have $\int \sqrt{a^2 - x^2}\,dx$ we use $x = a\sin\theta$ (as above).
- If we have $\int \sqrt{a^2 + x^2}\,dx$ we use $x = a\tan\theta$.
- If we have $\int \sqrt{x^2 - a^2}\,dx$ we use $x = a\sec\theta$.

Example 2.20. Suppose we have $\displaystyle\int \frac{1}{x^2\sqrt{9+x^2}}\,dx$. Then we set $x = 3\tan\theta, dx = 3\sec^2\theta\,d\theta$ and have

$$\begin{aligned}\int \frac{1}{x^2\sqrt{9+x^2}}\,dx &= \int \frac{1}{9\tan^2\theta\sqrt{9(1+\tan^2\theta)}} \cdot 3\sec^2\theta\,d\theta \\ &= \int \frac{\sec^2\theta}{3\tan^2\theta\sqrt{9\sec^2\theta}}\,d\theta \\ &= \int \frac{\sec\theta}{9\tan^2\theta}\,d\theta = \int \frac{\cos\theta}{9\sin^2\theta}\,d\theta \\ &= \int \frac{1}{9u^2}\,du \qquad \text{where } u = \sin\theta,\ du = \cos\theta\,d\theta \\ &= -\frac{1}{9u} + C = \frac{-1}{9\sin\theta} + C = -(\csc\theta)/9 + C.\end{aligned}$$

Now we just need to figure out what $\csc\theta$ is. But we know $\tan\theta = x/3$, so we can draw a right triangle with an angle θ, where the opposite side has length x and the adjacent side has length 3, and thus the hypotenuse has length $\sqrt{x^2+9}$.



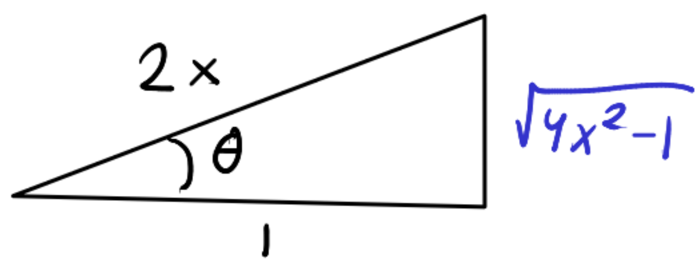
Then $\csc\theta = \sqrt{x^2+9}/x$ and thus we have

$$\int \frac{1}{x^2\sqrt{9+x^2}}\,dx = \frac{-\sqrt{x^2+9}}{9x} + C.$$

Example 2.21. Suppose we have $\displaystyle\int \frac{dx}{\sqrt{4x^2-1}}$. Then we can take $x = \frac{1}{2}\sec\theta$ and $dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$. We have

$$\begin{aligned}\int \frac{dx}{\sqrt{4x^2-1}} &= \int \frac{\frac{1}{2}\sec\theta\tan\theta\,d\theta}{\sqrt{\sec^2\theta-1}} \\ &= \frac{1}{2}\int \frac{\sec\theta\tan\theta\,d\theta}{\sqrt{\tan^2\theta}} \\ &= \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta+\tan\theta| + C.\end{aligned}$$

We know that $\sec\theta = 2x$ by our definition of θ. To find $\tan\theta$ we draw a triangle: angle θ has hypotenuse $2x$ and adjacent side 1, and thus opposite side $\sqrt{4x^2-1}$, so $\tan\theta = \sqrt{4x^2-1}$. Thus



$$\int \frac{dx}{\sqrt{4x^2-1}}\,dx = \ln|x+\sqrt{4x^2-1}/2| + C.$$

2.3 Integration by Partial Fractions

A last major technique deals with irritating fractions. It's relatively straightforward to compute

$$\begin{aligned}\int \frac{1}{x-1}\,dx &= \ln|x-1| + C \\ \int \frac{x}{x^2-1}\,dx &= \frac{1}{2}\ln|x^2-1| + C,\end{aligned}$$

but it's considerably less clear how to integrate $\frac{2}{x^2-1}$. But if we somehow notice that $\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}$, then we have

$$\int \frac{2}{x^2-1}\,dx = \int \frac{1}{x-1} - \frac{1}{x+1}\,dx = \ln(x-1) - \ln(x+1) + C.$$

We want to find a way to break any fraction we have to deal with into simple fractions like those.

2.3.1 Polynomial Long Division

As a warmup, we need to remember (or learn for the first time) polynomial long division. Suppose we have a ratio of polynomials, and the numerator is higher degree than the denominator. We can split the ratio into a polynomial, plus a ratio where the numerator is smaller degree than the denominator.

Example 2.22. Consider $\dfrac{x^3+2x^2+1}{x+1}$. Looking at this term by term, we get

$$\begin{aligned}\frac{x^3+2x^2+1}{x+1} &= x^2 + \frac{x^2+1}{x+1}\\ &= x^2 + x + \frac{-x+1}{x+1}\\ &= x^2 + x - 1 + \frac{2}{x+1}\end{aligned}$$

and thus

$$\int \frac{x^3+2x^2+1}{x+1}\,dx = \int x^2 + x - 1 + \frac{2}{x+1}\,dx = \frac{x^3}{3} + \frac{x^2}{2} - x + 2\ln(x+1) + C.$$

Example 2.23. Suppose we want to compute $\int \frac{x^3+1}{x^2+1}\,dx$. We need to do a long division here:

$$\begin{aligned}x^3+1 &= x(x^2+1) - x + 1\\ \frac{x^3+1}{x^2+1} &= x - \frac{x}{x^2+1} + \frac{1}{x^2+1}\end{aligned}$$

and thus

$$\begin{aligned}\int \frac{x^3+1}{x^2+1} &= \int x - \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx\\ &= \frac{x^2}{2} - \frac{1}{2}\ln|x^2+1| + \arctan(x) + C.\end{aligned}$$

2.3.2 Partial Fraction Decomposition

Once we have a fraction where the numerator is lower degree than the denominator, we factor the denominator and pull the fraction apart. By the Fundamental Theorem of Algebra, we can always factor any polynomial into a product of linear and quadratic factors—that is, degree-1 and degree-2 polynomials.

If we are asked to integrate a rational function $\frac{P(x)}{Q(x)}$, we begin by factoring Q completely into a product of linear and quadratic polynomials. We try to write $\frac{P(x)}{Q(x)}$ as a sum of fractions whose denominators are distinct factors of Q.

Example 2.24. Suppose we wish to find $\int \frac{3x^2-1}{x^3-x}\,dx$. We note that the numerator is smaller in degree than the denominator, so we don't have to do long division. We see that the denominator factors into $x(x+1)(x-1)$. So we wish to solve the equation

$$\begin{aligned} \frac{3x^2-1}{x^3-x} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} \\ 3x^2 - 1 &= A(x^2-1) + B(x^2-x) + C(x^2+x) \end{aligned}$$

From this point, there are two different approaches you can take. One is to group like terms together and then get a system of equations to solve. So we get

$$3x^2 - 1 = (A+B+C)x^2 + (C-B)x - A.$$

For two polynomials to be equal, each of their coefficients need to be equal; so we get a system of equations

$$\begin{aligned} 3 &= A+B+C \\ 0 &= C-B \\ -1 &= -A. \end{aligned}$$

The third equation tells us that $A = 1$, and the second equation tells us that $C = B$; from that the first equation tells us that $2 = B + C = 2B$ and thus $B = C = 1$. So we can write

$$\begin{aligned} \int \frac{3x^2-1}{x^3-x}\,dx &= \int \frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-1}\,dx \\ &= \ln|x| + \ln|x+1| + \ln|x-1| + C. \end{aligned}$$

If you take Linear Algebra (Math 2184 or 2185) you will learn a systematic approach to solving systems of equations like this.

However, there's a usually-simpler way of approaching this. If we look back at our first equation

$$3x^2 - 1 = A(x^2 - 1) + B(x^2 - x) + C(x^2 + x)$$

we can try plugging in numbers for x. For instance, if we set $x = 0$ we get

$$-1 = A(0 - 1) + B(0 - 0) + C(0 + 0) = -A$$

and thus $A = 1$. Similarly we can plug in $x = 1$ to get $2 = 2C$, or $x = -1$ to get $2 = 2B$.

How did we pick these values for x? These are precisely the roots of $x(x + 1)(x - 1)$; they're the places where we'd be dividing by zero in the original denominator. So one more way to think about this is that we take our original equation and multiply through by the denominator of just one term:

$$\begin{aligned} \frac{3x^2 - 1}{x^3 - x} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} \\ \frac{3x^2 - 1}{x^2 - 1} &= A + \frac{Bx}{x+1} + \frac{Cx}{x-1}. \end{aligned}$$

Now we see that plugging in $x = 0$ will no longer cause division-by-zero errors, but it will kill off everything on the right-hand side that didn't originally have an x term in the denominator.

If we have repeated factors in the denominator, things are a bit trickier. We need to have one fraction for each possible power of each linear factor. For instance, if we wish to integrate $\frac{1}{x^3(x-1)^3}$ we could write

$$\frac{1}{x^3(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2} + \frac{F}{(x-1)^3}.$$

Looking at this formula, we may decide that we do not, in fact, wish to integrate $\frac{1}{x^3(x-1)^3}$, and that this is why we have purchased a computer.

Example 2.25. If we have $\int \frac{2x+1}{x^3+2x^2+x}\,dx$, we see the bottom factors into $x(x + 1)^2$, with roots $-1, 0$. So we write

$$\begin{aligned} \frac{2x + 1}{x^3 + 2x^2 + x} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \\ 2x + 1 &= A(x^2 + 2x + 1) + B(x^2 + x) + C(x). \end{aligned}$$

If we plug in $x = 0$ we get $1 = A$. If we plug in -1 we get $-1 = -C$ so $C = 1$.

But now we've run out of roots; how do we figure out what B is supposed to be? Well, we now know what A and C are, so we have

$$2x + 1 = x^2 + 3x + 1 + B(x^2 + x).$$

This still has to be true if we plug in any value for x, we can just pick our favorite value. I'll pick $x = 1$ again, and we get $3 = 5 + 2B$ and thus $B = -1$. So we have

$$\begin{aligned} \int \frac{2x+1}{x^3 + 2x^2 + x}\, dx &= \int \frac{1}{x} - \frac{1}{x+1} + \frac{1}{(x+1)^2}\, dx \\ &= \ln|x| - \ln|x+1| - \frac{1}{x+1} + C. \end{aligned}$$

Sometimes we're stuck with quadratic factors. We treat them the same way we did the linear factors, except now our numerators will have terms like $Ax + B$ instead of just solitary numbers.

Example 2.26. If we wish to find $\int \frac{3x-1}{x(x^2+1)}\, dx$, we write

$$\begin{aligned} \frac{3x-1}{x(x^2+1)} &= \frac{A}{x} + \frac{Bx+C}{x^2+1} \\ 3x - 1 &= A(x^2+1) + (Bx+C)x \end{aligned}$$

Here again, setting $x = 0$ gives that $-1 = A$; since we're out of roots, we maybe pick some other numbers to plug in. If $x = 1$ we get $2 = 2A + B + C$, and $A = -1$, so $B + C = 4$. If $x = 2$ then $5 = 5A + 4B + 2C$ so $2B + C = 5$. Combining these two equations gives us that $B = 1$ and thus $C = 3$. Thus

$$\int \frac{3x-1}{x(x^2+1)}\, dx = \int \frac{-1}{x} + \frac{x+3}{x^2+1}\, dx.$$

That last term looks hard to deal with, but becomes easier if we split the numerator up into two pieces. One becomes a derivative of arctan, and the other becomes a u-substitution.

$$\begin{aligned} \int \frac{3x-1}{x(x^2+1)}\, dx &= \int \frac{-1}{x} + \frac{x}{x^2+1} + \frac{3}{x^2+1}\, dx \\ &= -\ln|x| + \frac{1}{2}\ln|x^2+1| + 3\arctan(x) + C. \end{aligned}$$

Remark 2.27. Arguably, we weren't out of roots for $x^3 + x$ there; if we allow complex numbers we could take $i = \sqrt{-1}$ as a root. Plugging that in to the equation would give

$$\begin{aligned} 3i - 1 &= A(-1+1) + (Bi + C)i \\ 3i - 1 &= -B + Ci \end{aligned}$$

and thus $C = 3$ and $B = 1$. In principle this is the more "correct" way to do this problem, but I'm not going to expect you to work with complex numbers.

Example 2.28. Consider $\int \frac{2x^2+10x+13}{x(x^2+6x+13)}\,dx$. We write

$$\begin{aligned}\frac{2x^2+10x+13}{x(x^2+6x+13)} &= \frac{A}{x} + \frac{Bx+C}{x^2+6x+13} \\ 2x^2+10x+13 &= A(x^2+6x+13) + (Bx+C)x.\end{aligned}$$

Plugging in $x = 0$ gives $A = 1$, leaving us with

$$x^2 + 4x = Bx^2 + Cx$$

and so $C = 4, B = 1$. So we have

$$\int \frac{2x^2+10x+13}{x(x^2+6x+13)}\,dx = \int \frac{1}{x} + \frac{x+4}{x^2+6x+13}\,dx.$$

The first bit is easy, but the second bit is tricky; we need to find a u we can substitute in.

Our life is much easier if the denominator is a sum of squares, so we try to write it that way by completing the square. We notice that $x^2 + 6x + 9 = (x + 3)^2$, so the denominator is $(x + 3)^2 + 4$; we try $u = x + 3, du = dx$. Then

$$\begin{aligned}\int \frac{x+4}{x^2+6x+13}\,dx &= \int \frac{u+1}{u^2+4}\,du \\ &= \int \frac{u}{u^2+4}\,du + \int \frac{1}{u^2+4}\,du \\ &= \frac{1}{2}\ln(u^2+4) + \frac{1}{2}\arctan(u/2) + C \\ &= \frac{1}{2}\left(\ln(x^2+6x+13) + \arctan\left(\frac{x+3}{2}\right)\right) + C \\ \int \frac{2x^2+10x+13}{x(x^2+6x+13)}\,dx &= \ln|x| + \frac{1}{2}\left(\ln(x^2+6x+13) + \arctan\left(\frac{x+3}{2}\right)\right) + C.\end{aligned}$$

Example 2.29. Consider $\int \frac{x^4+x^3+4x^2+x+1}{x(x^2+1)^2}\,dx$. We compute

$$\begin{aligned}\frac{x^4+x^3+4x^2+x+1}{x(x^2+1)^2} &= \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \\ x^4+x^3+4x2+x+1 &= A(x^2+1)^2 + (Bx+C)(x^3+x) + (Dx+E)x \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A\end{aligned}$$

So we see quickly that $A = 1$ and thus $B = 0$. Similarly, $C = 1$. This tells us that $E = 0$ and $D = 2$. Then we have

$$\int \frac{x^4 + x^3 + 4x^2 + 2x + 1}{x(x^2+1)} \, dx = \int \frac{1}{x} + \frac{1}{x^2+1} + \frac{2x}{(x^2+1)^2} \, dx$$
$$= \ln|x| + \arctan(x) - (x^2+1)^{-1} + C.$$

And finally, sometimes we have to combine all this with polynomial long division.

Example 2.30. Consider $\int \frac{x^3 + x^2 + 3x + 1}{x^2 + x} \, dx$.

We see that the numerator has higher degree than the denominator, so we should start by doing a polynomial long division. We work out that

$$x^3 + x^2 + 3x + 1 = (x^2 + x)(x) + 3x + 1$$
$$\frac{x^3 + x^2 + 3x + 1}{x^2 + x} = x + \frac{3x+1}{x^2+x}.$$

Now we can do a partial fractions decomposition

$$\frac{3x+1}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}$$
$$3x + 1 = A(x+1) + Bx.$$

Plugging in 0 gives $A = 1$, and plugging in -1 gives $-2 = -B$ or $B = 2$. Thus we have

$$\int \frac{x^3 + x^2 + 3x + 1}{x^2 + x} \, dx = \int x + \frac{1}{x} + \frac{2}{x+1} \, dx$$
$$= \frac{x^2}{2} + \ln|x| + 2\ln|x+1| + C.$$

A Brief Note on How to Cheat

We've developed a lot of techniques for evaluating integrals over the past couple of weeks. However, as good mathematicians we're also fundamentally lazy and would prefer to avoid work when we can manage it. There are two common solutions here.

First, your textbook has an extensive integral table, and even more extensive tables can be found online. It often requires some massaging to get your integral into the form of the table, but for complex integrals the table will be much easier than figuring things out from scratch. (For instance, the table incorporates the results of trig subsitution without making you work through it explicitly).

Second, computers are very good at doing integrals. Wolfram Alpha can often integrate a function for you, as can other computer tools. It's dangerous to become overly reliant on these tools—it's easy to make a mistake if you don't understand what's going on, and sometimes the computer will return the answer in a less useful form. They are very good for automated computations and checking your work, however.

A final cautionary note: there are some functions that don't have a nice closed-form antiderivative. Famously, there's no way to write $\int e^{x^2}\,dx$ in terms of "elementary functions." That doesn't mean there is no antiderivative; the obvious one is $\int_0^x e^{t^2}\,dt$. But while correct, that answer isn't terribly enlightening.

However, some of these non-antidifferentiable functions are very important. For instance, $f(x) = e^{-x^2}$ is the "bell curve" or "normal distribution" that is critical to statistics. We cannot antidifferentiate this, but we can still do some useful computations. Later on in section 5 we'll see how we can use "infinite series" to get a handle on this sort of function.

But right now we can take another approach: we can try to get an approximate answer to questions that we cannot solve exactly.

2.4 Numeric Integration

Sometimes we have a function which we, for some reason, can't compute an exact antiderivative of: either it is too difficult, or none exists, or we simply don't have enough data because we are using experimental measurements. In these cases we can use numerical methods to approximate the integral of a function.

In Calculus I we used the basic Riemann sum, generally defaulting to using the right endpoint as the sample point:

$$\int_a^b f(t)\,dt \approx R_n = \sum_{i=1}^n \Delta x \cdot f(x_i) \qquad \Delta x = \frac{b-a}{n}.$$

This is generally a pretty good approximation, but it can fail very badly if the right endpoints aren't representative samples of the function output. We can improve this easily, if slightly, by sampling at better locations: we lose slightly less information if we sample at the **midpoint** of each interval. If we're taking a limit, this doesn't matter, which is why we didn't worry about it in Calc 1. But when we're trying to get numeric estimates out of the Riemann sums, it does matter.

Definition 2.31. We can define the *midpoint approximation with n rectangles* of an integral to be

$$\int_a^b f(t)\,dt \approx M_n = \sum_{i=1}^n \Delta x \cdot f\left(\frac{x_i + x_{i-1}}{2}\right).$$

Example 2.32. Let's approximate $\int_0^4 4x^3\,dx$ using four intervals. With right endpoints we have

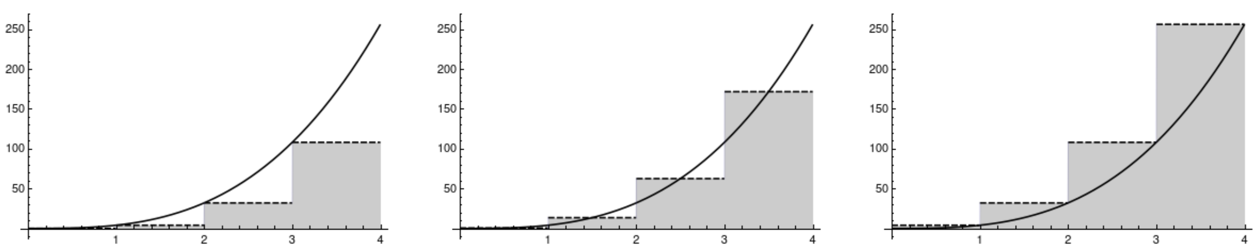
$$R_4 = \frac{4}{4}\left(f(1) + f(2) + f(3) + f(4)\right) = 4 + 32 + 108 + 256 = 400.$$

If we use midpoints instead we get

$$\begin{aligned} M_4 &= \frac{4}{4}\left(f\left(\frac{0+1}{2}\right) + f\left(\frac{1+2}{2}\right) + f\left(\frac{2+3}{2}\right) + f\left(\frac{3+4}{2}\right)\right) \\ &= .5 + 13.5 + 62.5 + 171.5 = 248. \end{aligned}$$

(The true answer, of course, is $x^4|_0^4 = 4^4 = 256.$)

Figure 2.1: The left endpoint, midpoint, and right endpoint Riemann sums for $\int_0^4 4x^3\,dx$. Notice how the midpoint sum visually has much less error than the other two.



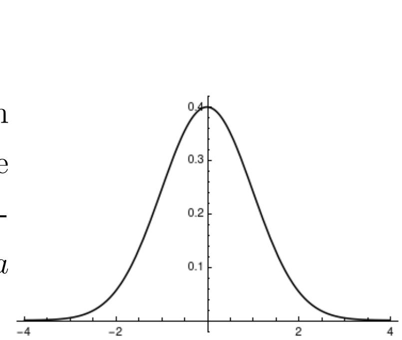
We might want to know how much error we can still have in this approximation. In general, if the function is jumping up and down wildly, we could have a ton of error; but if the function is tamer, we can keep the error fairly small.

Specifically, the extent to which the function can jump around is limited by the second derivative. If we have a number K such that $|f''(x)| \leq K$ on the interval of integration, then the error must be less than or equal to $\dfrac{K(b-a)^3}{24n^2}$.

In example 2.32, the second derivative is $24x$, and as long as $0 \leq x \leq 4$ we have $24x \leq 96$, so we take $K = 96$. Then the rule tells us that our error will be less than $\dfrac{96 \cdot 4^3}{12 \cdot 4^2} = 16$. Since our actual error is 8, this is true.

Example 2.33.

The *standard normal distribution* is defined by the function $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. The graph of this function is the bell curve you've probably seen if you've ever done any statistics, with standard deviation 1. The probability of getting a result between a and b standard deviations away from 0 is $\int_a^b \phi(t)\,dt$.



This is something we often need while doing statistics. For instance, we may want to know what the odds of being within one standard deviation of the mean is; this number is equal to $\int_{-1}^{1} \phi(t)\, dt$. If we had an antiderivative $\Phi(x)$ then this would be easy, but we can't actually antidifferentiate $e^{-x^2/2}$. (This isn't just a lack of tools on our part! It's a theorem that we can't write down an antiderivative using reasonable symbols and formulas.)

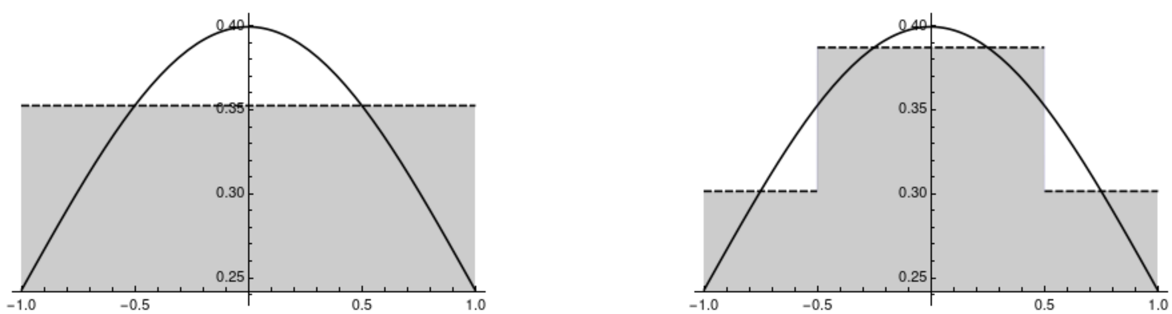
But this number is important, so we want an estimate of it anyway. We can use something like the midpoint rule to estimate it.

$$\int_{-1}^{1} \phi(t)\, dt \approx \phi(-.5) \cdot 1 + \phi(.5) \cdot 1 \approx .35 + .35 = .7$$

which should have relatively small error. But if we want it more accurate, we could do a midpoint approximation with more intervals:

$$\begin{aligned} \int_{-1}^{1} \phi(t)\, dt &\approx \phi(-.75) \cdot .5 + \phi(-.25) \cdot .5 + \phi(.25) \cdot .5 + \phi(.75) \cdot .5 \\ &\approx .30 \cdot .5 + .39 \cdot .5 + .39 \cdot .5 + .30 \cdot .5 = .69. \end{aligned}$$

Figure 2.2: Midpoint approximation to the normal distribution, with two and four intervals



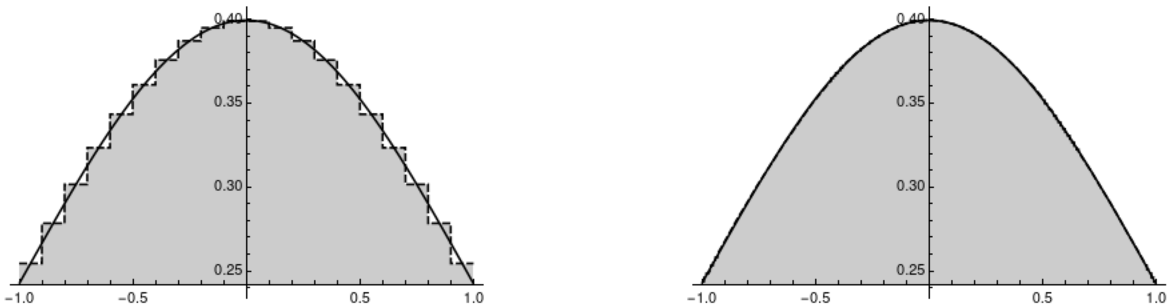
If we want a more precise estimate, we can always use more intervals. With twenty intervals, we get roughly .68—which is also what we get with 2000 intervals. Thus we can conclude there's a roughly 68% chance of landing within one standard deviation of the mean.

We can improve our approximations even more by moving away from rectangles altogether. Rather than sampling at one point, why not sample at *both* endpoints of the rectangle and average them? This leads to what is known as the **trapezoidal rule**:

$$\int_a^b f(t)\, dt \approx T_n = \sum_{i=1}^{n} \Delta x \cdot \frac{(f(x_{i-1}) + f(x_i))}{2}.$$

If you draw a picture it quickly becomes clear why this is a "trapezoidal" rule; we are taking the area of a trapezoid with base running from x_{i-1} to x_i and with top endpoints $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$.

Figure 2.3: Midpoint approximation to the normal distribution, with 20 and 200 intervals



This approximation has error $|E_T| \leq \dfrac{K(b-a)^3}{12n^2}$ if $|f''(x)| \leq K$. This is a worse error bound than the midpoint rule, but you can usually actually use more intervals with the trapezoid rule in practice.

Example 2.34. Let's approximate $\int_0^4 4x^3$ using four intervals, again. We get

$$\begin{aligned} T_4 &= \frac{4}{4}\left(\frac{f(0)+f(1)}{2} + \frac{f(1)+f(2)}{2} + \frac{f(2)+f(3)}{2} + \frac{f(3)+f(4)}{2}\right) \\ &= \frac{0+4}{2} + \frac{4+32}{2} + \frac{32+108}{2} + \frac{108+256}{2} = 272. \end{aligned}$$

Recall the true answer is 256, so we have error 16. This is no larger than the bound

$$E_T \leq \frac{96 \cdot 4^3}{12 \cdot 4^2} = 32.$$

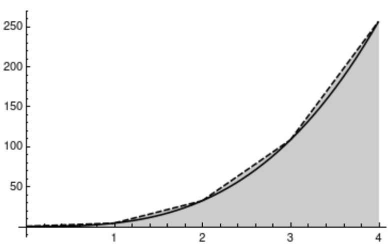


Figure 2.4: $\int_0^4 4x^3\,dx$ approximated with the trapezoid rule

Remark 2.35. There's a real sense in which the trapezoid rule isn't as good as the midpoint rule: we have $|E_M| \leq \dfrac{K(b-a)^3}{24n^2}$ while $|E_T| \leq \dfrac{K(b-a)^3}{12n^2}$, which is twice as big.

However, in a real sense, the midpoint rule requires twice as much *data* to work with—we need data on the midpoints, and not the endpoints, which generally means twice as many measurements. So the trapezoid rule can give better results in practice when working with real data.

We often want to use these techniques in real life when we're working from experimental data. Sometimes we have a bunch of specific measurements of the derivative, but we don't have an actual formula. Then we can't use the fundamental theorem of calculus to evaluate the integral exactly; but we can still approximate it from our data.

Example 2.36.

Suppose we have the speed of a runner at the following times:

$0s$	0	$.5s$	4.67	$1s$	7.34	$1.5s$	8.86	$2s$	9.73	$2.5s$	10.22
$3.0s$	10.51	$3.5s$	10.67	$4.0s$	10.76	$4.5s$	10.81	$5.0s$	10.81		

Can we estimate the distance covered?

This is in fact an integral: we're giving data about the velocity, or derivative, and we want to know about the distance, which is the original function. So we want to compute $\int_0^5 v(t)\,dt$. We can't possibly do a "real" integral because we don't have a formula for the whole function, but we can use the data we collected to estimate the integral.

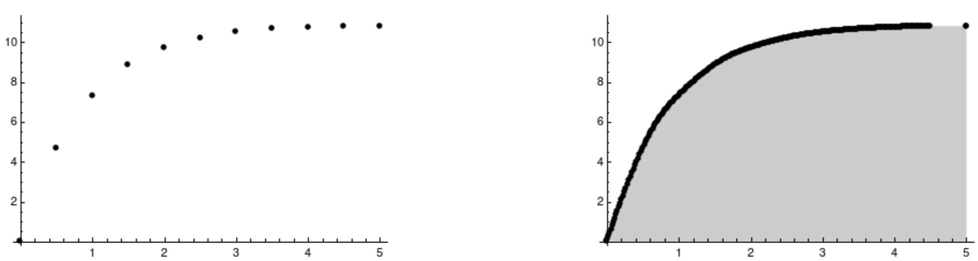


Figure 2.5: Left: Measurements of the runner's speed. Right: the shaded region is the distance covered

We can estimate using the Trapezoid rule:

$$\begin{aligned}
T_{10} = \frac{1}{2}\Bigg(&\frac{0+4.67}{2} + \frac{4.67+7.34}{2} + \frac{7.34+8.86}{2} + \frac{8.86+9.73}{2} + \frac{9.73+10.22}{2} + \frac{10.22+10.51}{2} \\
&+ \frac{10.51+10.67}{2} + \frac{10.67+10.76}{2} + \frac{10.76+10.81}{2} + \frac{10.81+10.81}{2}\Bigg) \\
&= \frac{1}{2}\left(2.335 + 6.005 + 8.1 + 9.295 + 9.975 + 10.365 + 10.59 + 10.715 + 10.785 + 10.81\right) \\
&= 44.4875.
\end{aligned}$$

In contrast, if we want to use the midpoint approximation, we can only use five intervals.

$$M_5 = 1 \cdot \big(4.67 + 8.86 + 10.22 + 10.67 + 10.81\big) = 45.21$$

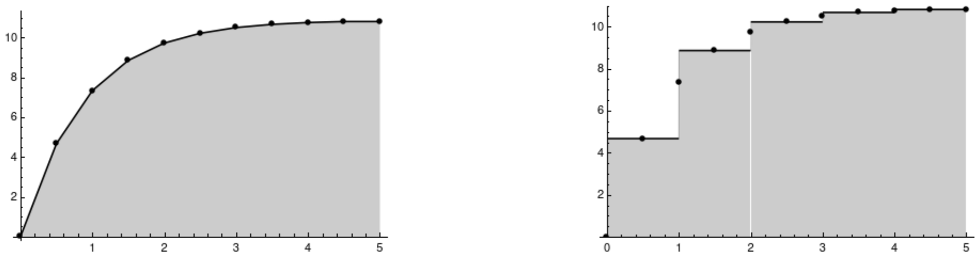


Figure 2.6: Left: an approximation with the trapezoid rule and ten intervals. Right: an approximation with the midpoint rule and five intervals.

If averaging two points is good, then averaging three points must be better, right? Rather than sampling one point, or making a trapezoid out of each pair of points, we can draw a parabola through each set of three points. A bit of algebra gives **Simpson's Rule**:

$$\begin{aligned}\int_a^b f(x)\,dx \approx S_n &= \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)\\ &= \frac{\Delta x}{3}\left(f(x_0) - f(x_n) + \sum_{i=1}^{n/2} 4f(x_{2i-1}) + 2f(x_{2i})\right).\end{aligned}$$

(Note that this assumes n is even).

If $|f^{(4)}(x)| \leq L$ for $a \leq x \leq b$, and E_S is the error in Simpson's rule, then

$$|E_S| \leq \frac{L(b-a)^5}{180n^4}.$$

Example 2.37. What happens if we estimate our runner's speed with Simpson's rule? We get

$$\begin{aligned}S_{10} &= \frac{1}{6}\left(0 + 4\cdot 4.67 + 2\cdot 7.34 + 4\cdot 8.86 + 2\cdot 9.73 + 4\cdot 10.22\right.\\ &\qquad \left.+2\cdot 10.51 + 4\cdot 10.67 + 2\cdot 10.76 + 4\cdot 10.81 + 10.81\right)\\ &= \frac{268.41}{6} = 44.735.\end{aligned}$$

If averaging three points is better, then averaging four must be even better, right? Well, technically, yes, but it's almost never worth the effort.

Example 2.38. Suppose we want to compute $\int_0^2 e^{x^2}$. How many intervals do we need, with each method, to guarantee the error is less than 1 in a thousand–that is, to guarantee the answer is correct to three decimal places?

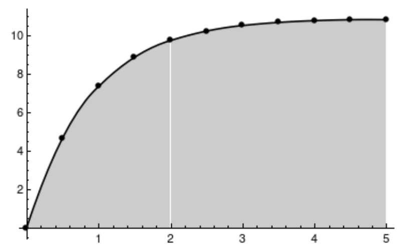


Figure 2.7: The runner's position estimated with Simpson's rule

On the interval, $f''(x) = 2e^{x^2} + 4x^2e^{x^2}$ is maximized when $x = 2$. $f''(2) = 18e^4 = K$. So we have

$$|E_M| \leq \frac{K \cdot 2^3}{24n^2} = \frac{6e^4}{n^2}$$
$$|E_T| \leq \frac{K \cdot 2^3}{12n^2} = \frac{12e^4}{n^2}.$$

Thus the trapezoid approximation is guaranteed to be accurate to within 1/1000 when $n > 809$, and the midpoint approximation is guaranteed to be accurate to within 1/100 when $n > 572$.

$f''''(x)$ is maximized at $x = 2$, where it is equal to $460e^4 = L$. Thus

$$|E_S| \leq \frac{L \cdot 2^5}{180n^4} = \frac{736e^4}{9n^4}.$$

Thus the Simpson's rule approximation is guaranteed to be accurate to within 1/1000 when $n > 45$.

So to get an answer to within 1/1000 we can use a computer to compute

$$S_{46} = \frac{2}{46 \cdot 3}\left(f(0) - f(2) + \sum_{i=1}^{23} 4f\left(\frac{2(2i-1)}{46}\right) + 2f\left(\frac{2 \cdot 2i}{46}\right)\right) = \frac{2 \cdot 1135.24}{46 \cdot 3} = 16.4528.$$

Note that since the true answer is 16.4526, this is accurate within $2/10,000$, which is what we wanted.

3 Applications of the Integral

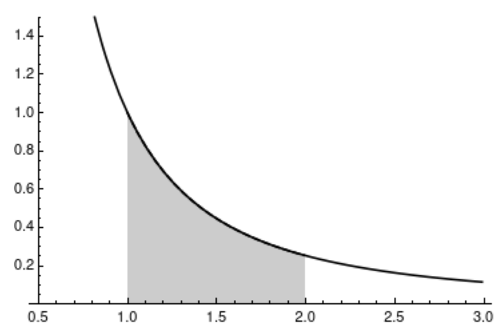
Now that we've learned some techniques for doing more complicated integrals, we want to use them to accomplish something. Why do we want to compute integrals, and what can they do for us?

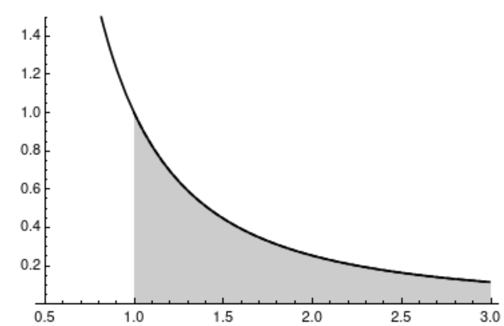
3.1 Improper Integrals and Unbounded Area

Recall that integrals were originally defined to compute areas. But so far we've only looked at the areas of regions that are, in some sense, "bounded": we may need calculus to find the exact area of the region, but we know the area is finite (and thus is a number)because we can draw a big circle around the whole shape. But sometimes we want to find the area of shapes that extend infinitely in one direction.

Example 3.1 (Motivating Example)**.** First off, we can ask: what is the area of the region bounded by the x-axis, the lines $x = 1$ and $x = 2$, and the curve $y = 1/x^2$? This is the sort of question we asked a lot in calculus 1. We can compute this area as

$$\int_1^2 \frac{dx}{x^2} = \frac{-1}{x}\Big|_1^2 = \frac{-1}{2} - \frac{-1}{1} = \frac{1}{2}.$$





But we can ask a trickier question. What is the area of the region bounded solely by the x-axis, the line $x = 1$, and the curve $y = 1/x^2$? Notice this region doesn't have any boundary at all on the right edge.

At first we don't know what to do, since our integrals are only defined on finite intervals. But we imagine the "remaining" area of the region must get smaller and smaller as x gets bigger and bigger. So what happens if we take the integral of a big chunk of the region?

If we look at the region bounded by $x = 1$ and $x = N$, for some number N, we get the area

$$A_N = \int_1^N \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^N = \frac{-1}{N} - \frac{-1}{1} = 1 - \frac{1}{N}.$$

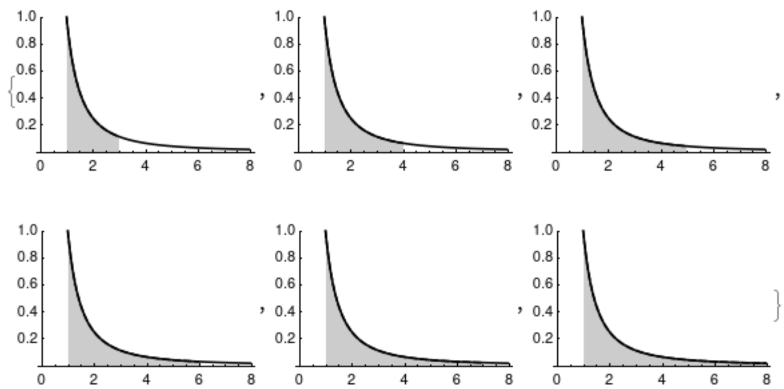


Figure 3.1: A sequence of finite, proper integrals covering increasing amounts of the area under the curve.

As N gets very large, we see that this area approaches 1, so we conclude the area of the whole region is 1.

There are two different ways for regions to be unbounded; it's entirely possible for both to happen at once, but we can always separate them and deal with them separately. We call such integrals *improper integrals*.

3.1.1 Improper integrals to ∞

The first situation is the situation in our motivating example, where we have to integrate over an "infinitely wide region."

Definition 3.2. If $\int_a^t f(x)\,dx$ exists for every $t \geq a$, then we define the *improper integral*

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx$$

provided this limit exists.

If $\int_t^b f(x)\,dx$ exists for every $t \leq b$, we define

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided the limit exists.

We say these integrals are *convergent* if the limit exists and *divergent* if the limit does not exist.

If both integrals are convergent, we write

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{+\infty} f(x)\,dx.$$

In this case it doesn't matter which a we use: either both integrals will converge for every a, or one won't converge for any a. If one or both integrals don't converge, then $\int_{-\infty}^{+\infty} f(x)\,dx$ doesn't have a clear meaning and we should avoid writing it.

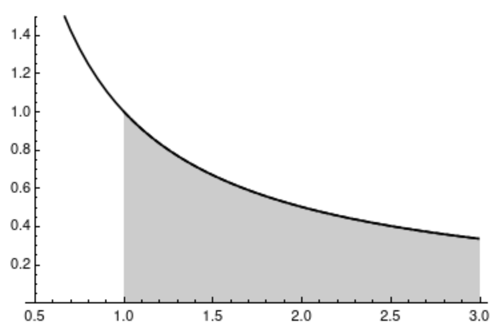
Remark 3.3. In this language, example 3.1 calculated that

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = 1.$$

Example 3.4. What is $\int_1^{+\infty} \frac{1}{x}\,dx$?

$$\begin{aligned}\int_1^{+\infty} \frac{1}{x}\,dx &= \lim_{t\to+\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to+\infty} (\ln|x|)\big|_1^t \\ &= \lim_{t\to+\infty} (\ln|t| - \ln|1|) = \lim_{t\to+\infty} \ln|t| = +\infty.\end{aligned}$$

Thus this integral is divergent. Geometrically, this means that the area under this curve is in fact infinite.



Remark 3.5. It turns out that $\int_1^{+\infty} x^r\,dx$ is convergent whenever $r < -1$ and divergent whenever $r \geq -1$. This is worked out in your textbook, and will have important consequences in section 4.3.

Example 3.6. What is $\int_{-\infty}^{\pi} \sin(x)\,dx$?

We write this as a limit:

$$\begin{aligned}\int_{-\infty}^{\pi} \sin(x)\,dx &= \lim_{t\to-\infty} \int_t^{\pi} \sin(x)\,dx \\ &= \lim_{t\to-\infty} (-\cos(x))\big|_t^{\pi} \\ &= \lim_{t\to-\infty} (-\cos(\pi) - (-\cos(t))) = \lim_{t\to-\infty} \cos(t) + 1.\end{aligned}$$

This limit does not exist (because $\cos(x)$ is periodic), so the integral is divergent. Note that in this case the (net) area isn't infinite; it just isn't well-defined.

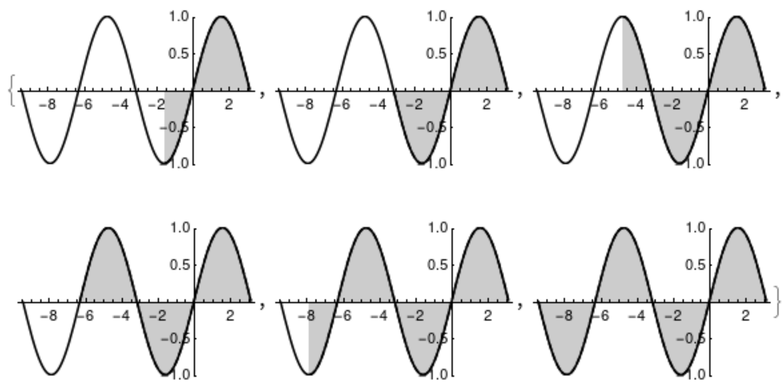
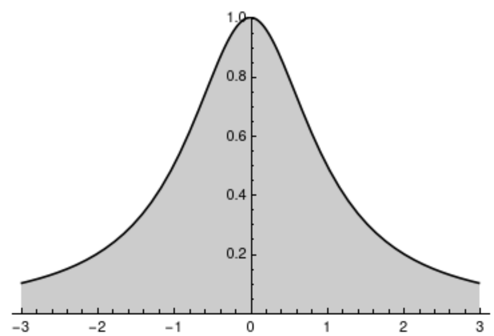


Figure 3.2: A sequence of finite integrals of $\cos(x)$ Notice that the net area is neither always increasing, nor always decreasing, nor trending to a specific number.

Example 3.7.



What is $\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, dx$?

First think about what you expect to happen. The graph of this function peaks in the middle at $(0, 1)$, and trails off to zero as x gets large or small. So it's plausible that this integral is finite. It is certainly positive.

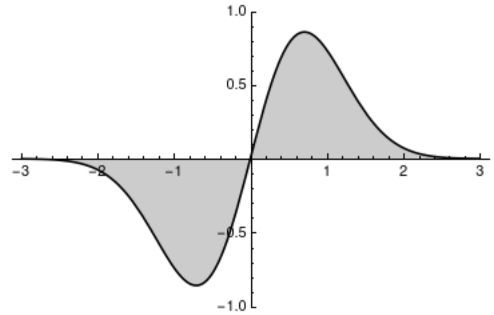
We can pick any a we want, and it's convenient to pick $a = 0$ so things are symmetrical.

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} &= \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2} \\
&= \lim_{t\to-\infty} \int_t^0 \frac{dx}{1+x^2} + \lim_{s\to+\infty} \int_0^s \frac{dx}{1+x^2} \\
&= \lim_{t\to-\infty} \arctan(x)|_t^0 + \lim_{s\to+\infty} \arctan(x)|_0^s \\
&= \lim_{t\to-\infty} (\arctan(0) - \arctan(t)) + \lim_{s\to+\infty} (\arctan(s) - \arctan(0)) \\
&= -\lim_{t\to-\infty} \arctan(t) + \lim_{s\to+\infty} \arctan(s) = -(-\pi/2) + \pi/2 = \pi.
\end{aligned}$$

Both partial integrals are convergent, so the total integral is convergent and the area under the curve is π.

Example 3.8. $\int_{-\infty}^{+\infty} 2xe^{-x^2}\, dx$.

We can look at a graph, and see that the areas on either side appear to balance each other out; we might expect the integral to be zero. But it's important to notice that this *only* works if each side has a finite area—the way we defined things, we can't have two infinite regions balancing each other out.



(It's possible to handle that case sensibly but you have to be a lot more careful definiting exactly what limit you're taking; in general it's a $\infty - \infty$ indeterminate form.)
We again split this into two integrals. We will also set $u = x^2, du = 2x\,dx$.

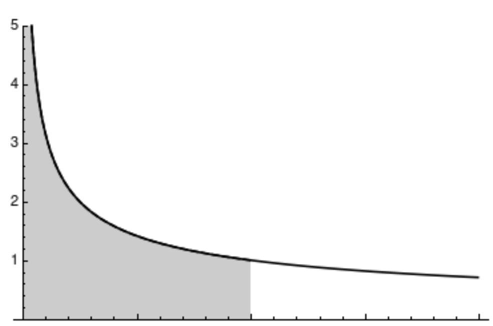
$$\begin{aligned}
\int_{-\infty}^{+\infty} 2xe^{-x^2}\,dx &= \lim_{t\to-\infty}\int_t^0 2xe^{-x^2}\,dx + \lim_{s\to+\infty}\int_0^s 2xe^{-x^2}\,dx \\
&= \lim_{t\to-\infty}\int_{t^2}^0 e^{-u}\,du + \lim_{s\to+\infty}\int_0^{s^2} e^{-u}\,du \\
&= \lim_{t\to-\infty} -e^{-u}|_{t^2}^0 + \lim_{s\to+\infty} -e^{-u}|_0^{s^2} \\
&= \lim_{t\to-\infty} -e^0 - (-e^{-t^2}) + \lim_{s\to+\infty} -e^{-s^2} - (-e^0) = -1 - 0 + 0 + 1 = 0.
\end{aligned}$$

3.1.2 Improper integrals of discontinuous functions

There's a completely separate type of problem, where our x-values are bounded but our function behaves badly somewhere in that interval. Generally the issue comes up when our region is infinite in the *vertical* direction.

Example 3.9 (Motivating Example)**.** What is the area under $f(x) = 1/\sqrt{x}$ between $x = 0$ and $x = 1$?

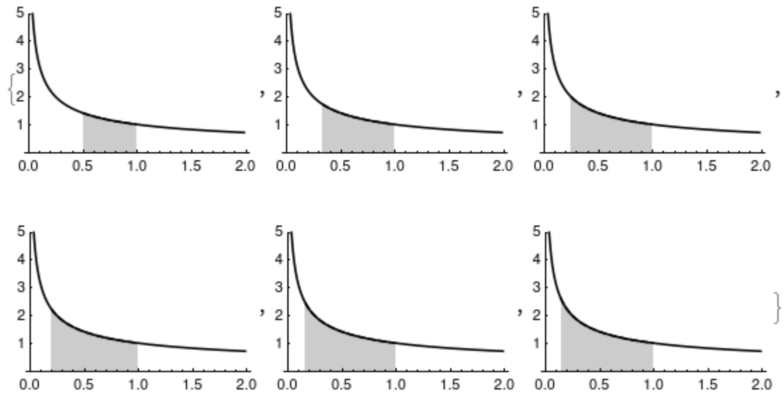
We can draw a clear picture of the region we want to study: But we can't use our normal



integral here because f isn't well-defined at 0. But we can find the area of almost all of the region, just as we did before. If ε is a small number, we have

$$\int_{\varepsilon}^{1} x^{-1/2}\,dx = 2x^{1/2}|_{\varepsilon}^{1} = 2(1-\sqrt{\varepsilon}).$$

It's easy to calculuate that $\lim_{\varepsilon\to 0} 2(1-\sqrt{\varepsilon}) = 2$, so we say the area of this region is 2.



Definition 3.10. If f is continuous on $[a,b)$ but discontinuous at b, we define the improper integral

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

provided the limit exists (and is finite).

If f is continuous on $(a,b]$ but discontinuous at a, we define

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

provided the limit exists.

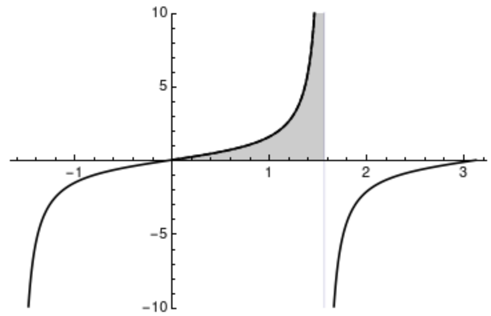
Again, the improper integral $\int_a^b f(x)\,dx$ is convergent if the limit exists, and divergent if it does not.

If f has a discontinuity at c for $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Remark 3.11. In this language, example 3.9 calculated that

$$\int_0^1 x^{-1/2}\,dx = 2.$$



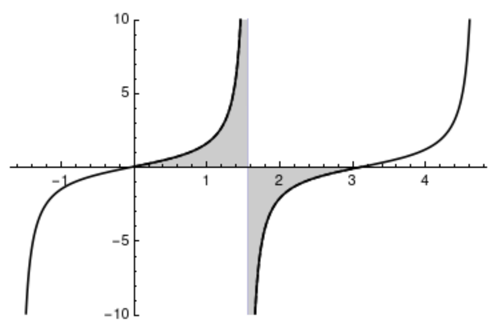
Example 3.12. What is $\int_0^{\pi/2} \tan x \, dx$?

$\tan(\pi/2)$ is not well defined. So we write

$$\begin{aligned}\int_0^{\pi/2} \tan x \, dx &= \lim_{t \to \pi/2^-} \int_0^t \tan x \, dx \\ &= \lim_{t \to \pi/2^-} \ln|\sec x| \Big|_0^t \\ &= \lim_{t \to \pi/2^-} \ln|\sec t| - \ln|1| = +\infty\end{aligned}$$

since $\lim_{t \to \pi/2^-} \sec t = +\infty$. So the integral is divergent.

Example 3.13 (Warning Example)**.** What is $\int_0^{\pi} \tan x \, dx$?



Again, it looks like the two infinities balance each other out, and there is a way to make a careful argument that justifies that impression. But we can't do it with the tools we have; this is again an $\infty - \infty$ indeterminate form.

If we're sloppy, we might reason as follows:

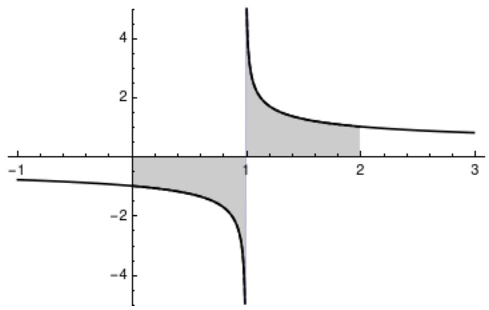
$$\int_0^{\pi} \tan x \, dx = \ln|\sec x| \Big|_0^{\pi} = \ln|-1| - \ln|1| = 0.$$

This is false because there is a discontinuity in the middle. We would need to split this integral into

$$\int_0^{\pi/2} \tan x \, dx + \int_{\pi/2}^{\pi} \tan x \, dx$$

and we already saw that the first integral is divergent (as is the second), so the whole integral is also divergent.

Example 3.14. What is $\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx$?



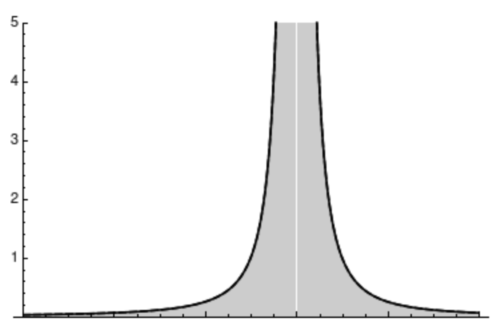
This is an improper integral since $\frac{1}{\sqrt[3]{x-1}}$ isn't defined at 1. Just like in example 3.13, it looks like the positive and negative components should balance each other.

To compute, we break it apart and compute a limit:

$$\begin{aligned}
\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx &= \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^2 \frac{1}{\sqrt[3]{x-1}}\,dx \\
&= \lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt[3]{x-1}}\,dx + \lim_{s\to 1^+} \int_s^2 \frac{1}{\sqrt[3]{x-1}}\,dx \\
&= \lim_{t\to 1^-} \left(\frac{3}{2}(x-1)^{2/3}\right)\Big|_0^t - \lim_{s\to 1^+} \left(\frac{3}{2}(x-1)^{2/3}\right)\Big|_s^2 \\
&= \lim_{t\to 1^-} \left(\frac{3}{2}(t-1)^{2/3} - \frac{3}{2}\cdot 1\right) + \lim_{s\to 1^+} \left(\frac{3}{2}\cdot 1 - \frac{3}{2}(s-1)^{2/3}\right) \\
&= 0 - 3/2 + 3/2 - 0 = 0.
\end{aligned}$$

And finally, sometimes we can have both varieties of impropriety in the same question.

Example 3.15. $\int_0^{+\infty} \frac{1}{(x-6)^2}\,dx$.



Looking carefully, we see this is improper in two ways. First, the upper bound of the integral is $+\infty$, as with the examples we saw in section 3.1.1. And second, the function has a vertical asymptote at $x = 6$. Thus we wind up needing to split this integral into *three* components:

$$\begin{aligned}\int_0^{+\infty} \frac{1}{(x-6)^2}\,dx &= \int_0^6 \frac{dx}{(x-6)^2} + \int_6^7 \frac{dx}{(x-6)^2} + \int_7^{+\infty} \frac{dx}{(x-6)^2} \\ &= \lim_{r\to 6^-} \int_0^r \frac{dx}{(x-6)^2} + \lim_{s\to 6^+} \int_s^7 \frac{dx}{(x-6)^2} + \lim_{t\to+\infty} \int_7^t \frac{dx}{(x-6)^2}.\end{aligned}$$

In order for our original integral to converge, we need all three of these to converge. But we see that

$$\begin{aligned}\lim_{r\to 6^-} \int_0^r \frac{dx}{(x-6)^2} &= \lim_{r\to 6^-} \left(\frac{-1}{x-6}\right)\Bigg|_0^r \\ &= \lim_{r\to 6^-} \frac{1}{6-r} - \frac{1}{6} = +\infty\end{aligned}$$

diverges, so the whole integral diverges.

3.1.3 The Comparison Test for Improper Integrals

Sometimes we don't care much what the area of a region is; we only want to know if it's finite or not. (This will come up again in section 4.3, but also comes up in many other applications.) In those cases this theorem is enough:

Theorem 3.16. *Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then:*

- *If $\int_a^{+\infty} f(x)\,dx$ is convergent then $\int_a^{+\infty} g(x)\,dx$ is convergent.*
- *If $\int_a^{+\infty} g(x)\,dx$ is divergent then $\int_a^{+\infty} f(x)$ is divergent.*

This basically tells us that if the area under $f(x)$ is finite, then any area it contains must be finite; and if the area under $g(x)$ is infinite, any area containing it must be infinite.

Example 3.17. $\int_1^{+\infty} x^r\,dx$ is convergent if $r \leq -2$ since $x^r \leq x^{-2}$ on $[1, +\infty)$, and we know $\int_1^{+\infty} x^{-2}$ converges from example 3.1.

$\int_1^{+\infty} x^r$ is divergent if $r \geq -1$ since $x^r \geq x^{-1}$ on $[1, +\infty)$ and we know that $\int_1^{+\infty} x^{-1}$ diverges from example 3.4.

Example 3.18. Does $\int_0^{+\infty} e^{-x^2}\,dx$ converge?

We will find this slightly easier if we split this up into two integrals.

It's clear that $\int_0^1 e^{-x^2}\,dx$ converges because it is a finite proper integral of a continuous function. So we just have to show that $\int_1^{+\infty} e^{-x^2}\,dx$ converges.

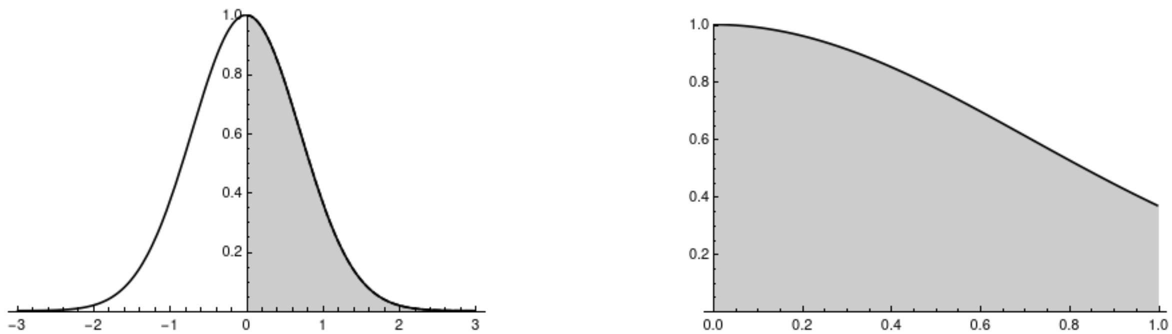


Figure 3.3: Left: the area computed by our improper integral $\int_0^{+\infty} e^{-x^2}$. Right: the proper integral $\int_0^1 e^{-x^2}$.

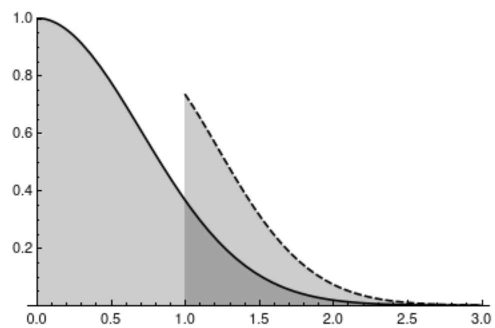


Figure 3.4: The function $2xe^{-x^2}$ is not always larger than e^{-x^2}. But when $x > 1$, the function $2xe^{-x^2}$ is in fact larger ,so we can use it with the comparison test.

But for every x in $[1, +\infty)$, we know that $e^{-x^2} < 2xe^{-x^2}$, and we've shown that $\int_1^{+\infty} 2xe^{-x^2}\,dx$ converges in example 3.8. Thus by the comparison test, $\int_1^{+\infty} e^{-x^2}\,dx$ converges.

It turns out that $\int_0^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}/2$; we can prove this using some clever tricks from complex analysis. But we don't really have the tools to compute it in this course.

3.2 Geometric Applications

We can also use the integral to answer some other fun little geometry questions.

3.2.1 Arc Length

One question we can ask is: given a curve, how long is it? If the "curve" is a straight line, this is easy. We can just use the Pythagorean Theorem.

Example 3.19. A line with endpoints $(1, 2)$ and $(4, 6)$ has an x-coordinate distance of 3, and a y-coordinate distance of 4. So the total length of this line segment is $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

If our curve is not a straight line, we have more trouble. But we can solve our problem by combining ideas from differential and integral calculus. Differential calculus tells us that, if we have a small window, we can approximate a function with a straight line—and we know how to compute the lengths of those. Integral calculus says that we can break a problem up into a bunch of pieces, solve each piece, and then bring them back together.

Suppose we have the graph of some function $f(x)$, as x runs from a to b. We can make a bad estimate of the lenth of this curve by just looking at the distance between the points $(a, f(a))$ and $(b, f(b))$. But if a and b are really close together, this estimate is actually pretty good; the function can't "curve away" from this line segment too much.

So what we can do is take the curve and split it up into a bunch of points that are close together, and try to approximate the length of each segment.

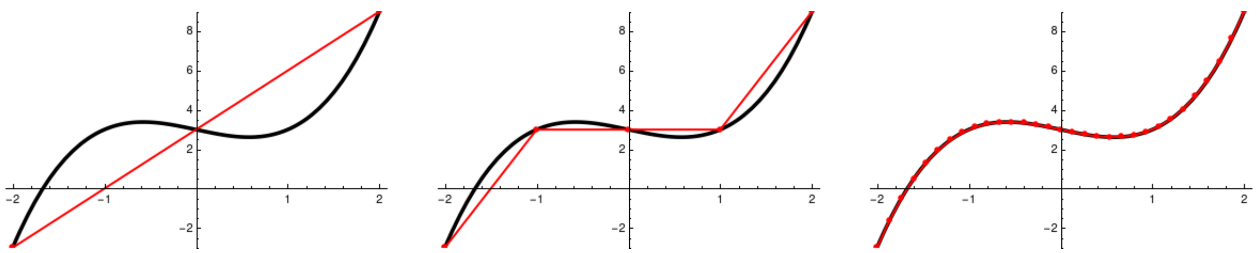


Figure 3.5: The curve $y = x^3 - x + 3$ approximated by one, four, and thirty segments

For finitely many segments, we could just work out the exact points individually. So we'd get something like $\sqrt{\Delta x^2 + (f(a + \Delta x) - f(a))^2}$ for the length of our first line segment, and a similar formula for each other line segment. In the middle picture above, we have $\Delta x = 1$ so our total formula would look like

$$\sqrt{1 + (f(-1) - f(-2))^2} + \sqrt{1 + (f(0) - f(-1))^2} + \sqrt{1 + (f(1) - f(0))^2} + \sqrt{1 + (f(2) - f(1))^2}.$$

This totally works to get an approximate value, and is the equivalent of the approach to integration we took in section 2.4.

But we want to take a limit as the number of pieces goes to infinity and Δx goes to 0. So we need some sort of nice formula. And this is where differential calculus comes to the rescue.

We know that near a point a, the function $f(x)$ is approximately equal to $f(a) + f'(a)(x - a)$. If we split the interval $[a, b]$ up into n sub-intervals, so that the ith interval goes from x_{i-1} to x_i, then the left-hand endpoint is $(x_{i-1}, f(x_{i-1}))$. The right-hand endpoint is *approximately*

$f(x_{i-1}) + f'(x_{i-1})(x_i - x_{i-1})$, so the length of the line between them is approximately

$$\begin{aligned}\sqrt{(x_i - x_{i-1})^2 + \Big(\big(f(x_{i-1}) + f'(x_{i-1})(x_i - x_{i-1})\big) - f(x_{i-1})\Big)^2} \\ = \sqrt{(x_i - x_{i-1})^2 + \Big(f'(x_{i-1})(x_i - x_{i-1})\Big)^2} \\ = \sqrt{(x_i - x_{i-1})^2(1 + f'(x_{i-1})^2)} = \Delta x \sqrt{1 + f'(x_{i-1})^2}.\end{aligned}$$

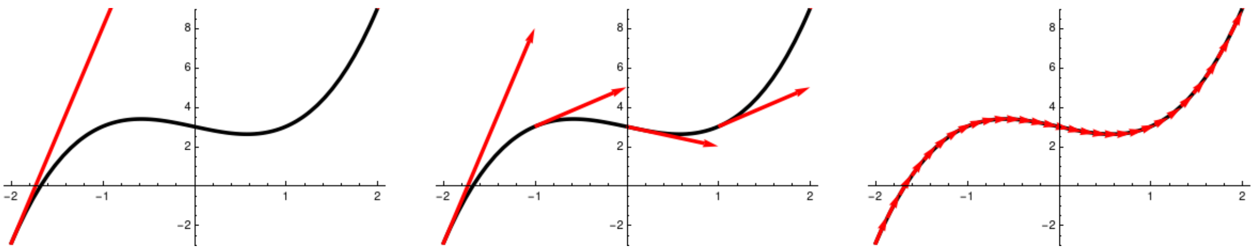


Figure 3.6: This derivative approximation is quite bad when $n = 1$, but as n gets larger we begin to get good approximations of the original curve.

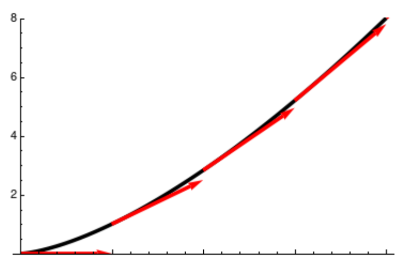
If we add up all these lengths, we get the formula

$$L \approx \sum_{i=1}^{n} \Delta x \sqrt{1 + f'(x_{i-1})^2}.$$

But this should look very familiar because this is just a Riemann sum—where the function is $\sqrt{1 + f'(x)^2}$. Thus we can take the limit as n goes to infinity and Δx goes to zero, and we get an exact formula for the length of the curve:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx \tag{1}$$

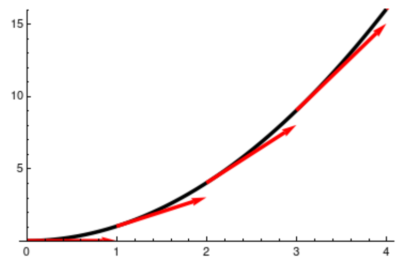
Example 3.20. Let's take the curve $y^2 = x^3$ and find the arc length of the curve between $(0, 0)$ and $(4, 8)$.



We have that $y = \sqrt{x^3}$ on this curve, so $y' = \frac{3}{2}x^{1/2}$. Then

$$\begin{aligned} L &= \int_0^4 \sqrt{1+(y')^2}\,dx \\ &= \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx = \frac{2}{3}\cdot\frac{4}{9}(1+9/4x)^{3/2}\Big|_0^4 \\ &= \frac{8}{27}\left(10^{3/2}-1\right). \end{aligned}$$

Example 3.21. Let $f(x) = x^2$. Let's find the arc length between $x = 0$ and $x = 4$.



We have

$$L = \int_0^4 \sqrt{1+(2x)^2}\,dx$$

This looks a lot like a trig sub integral. We can set $2x = \tan\theta$, so $dx = \frac{1}{2}\sec^2\theta\,d\theta$. When $x = 0$ we have $\tan\theta = 0$ so $\theta = 0$, and when $x = 4$ we have $\tan\theta = 8$ so $\theta = \arctan(8)$. This gives us

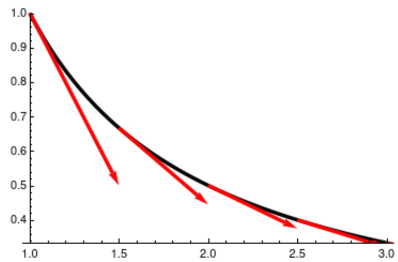
$$\begin{aligned} L &= \int_0^4 \frac{1}{2}\sqrt{1+(2x)^2}\,dx = \int_0^{\arctan 8} \frac{1}{2}\sqrt{1+\tan^2(\theta)}\sec^2\theta\,d\theta \\ &= \int_0^{\arctan 8} \frac{1}{2}\sec^3\theta\,d\theta \end{aligned}$$

and at this point I...give up on this integral. You can look it up, or you can plug it into a computer algebra package. We get

$$\frac{1}{4}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)|\Big|_0^{\arctan 8} \approx 16.819.$$

Often arc length calculations will give you pretty nasty integrals, and that's fine. The main goal here is to be able to set up these integrals—and, more importantly, to understand *why* this integral answers our question. On a quiz or a test, I will either pick questions that happen to give reasonable integrals, or tell you to just set up the integral and not actually compute it out.

Somtimes it's as easy—or easier—to integrate with respect to y.



Example 3.22 (Recitation)**.** Consider the graph of the hyperbola $xy = 1$ as y varies from 1 to 3. What is the arc length of this curve?

We could view this as a function of x: $y = 1/x$, so $y' = -1/x^2$, and then

$$L = \int_{1/3}^{1} \sqrt{1 + 1/x^4}\, dx \approx 2.14662.$$

Alternatively, we could view it as a function of y: $x = 1/y$, so $x' = -1/y^2$, and we have

$$L = \int_{1}^{3} \sqrt{1 + 1/y^4}\, dy \approx 2.14662.$$

Which approach is more convenient depends on what you're doing.

3.2.2 Surface Area

We can also kick this up a dimension. In Calc I, we computed the area under a curve, and then we computed the volume of a solid of revolution. Now we've computed the length of a curve; we can also compute the surface area of a surface produced by revolving a curve an axis. This is really just the area of the *outside* of the shapes we studied in Calculus I.

There are a couple of ways to think about this formula, but they both get you to essentially the same place. We can imagine cutitng the surface into little strips, and then pretending these strips are small cylinders. The radius of the cylinder is given by the height of the function; the height of the cylinder is given by the *arc length* of the bit of the function inside the band. (When the function is steeper, the average radius can be the same, but the width of our imaginary band is much greater, as we see in figure 3.7.)

Thus the area of one band will be the circumference of the circle, which is $2\pi f(x)$, times the width of the band, which from section 3.2.1 we know is approximately $\sqrt{1 + f'(x)^2}dx$. Thus the surface area of a surface of revolution is

$$SA = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx. \tag{2}$$

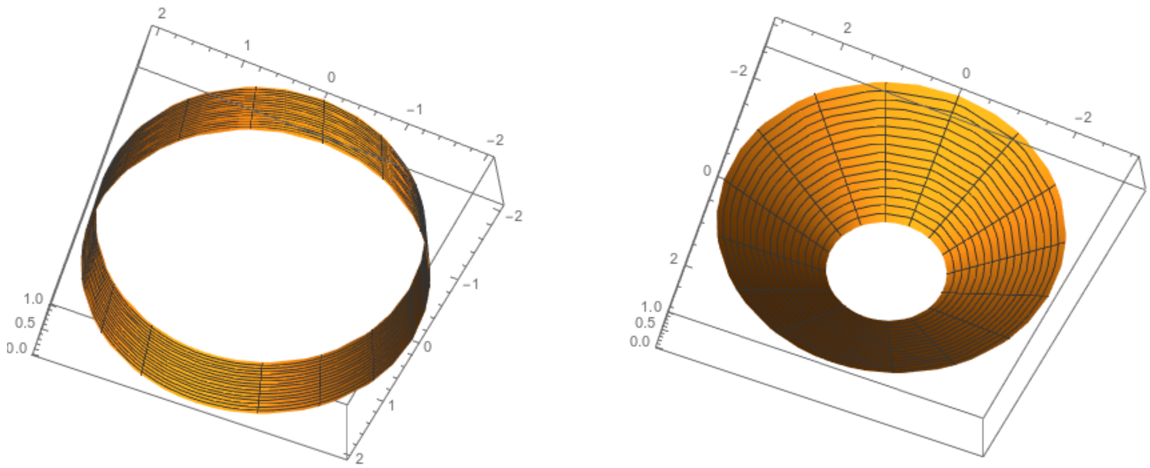


Figure 3.7: Both bands have the same x-axis thickness and the same average radius, but the right one has way more surface area

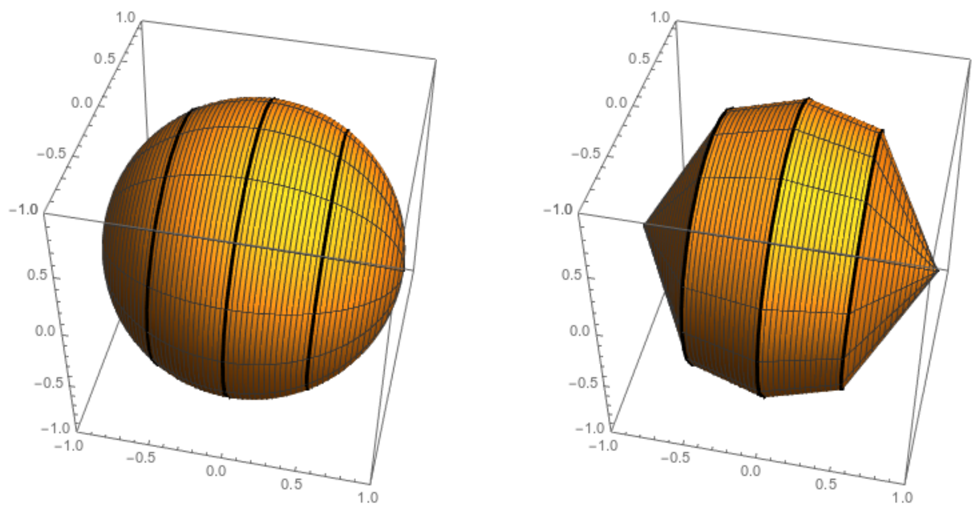


Figure 3.8: We can split the sphere into strips, and approximate each strip by pretending it's made of straight lines.

Example 3.23. What is the surface area of a sphere of radius 1? We can look at this as taking the curve $\sqrt{1-x^2}$ on $[-1,1]$ and revolving it around the x axis. Since $f'(x) = \frac{-x}{\sqrt{1-x^2}}$, we get

$$\begin{aligned}
SA &= \int_{-1}^{1} 2\pi\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}\,dx \\
&= 2\pi\int_{-1}^{1} \sqrt{1-x^2}\sqrt{\frac{1-x^2+x^2}{1-x^2}}\,dx \\
&= 2\pi\int_{-1}^{1} \sqrt{1-x^2}\sqrt{\frac{1}{1-x^2}}\,dx \\
&= 2\pi\int_{-1}^{1} 1\,dx = 4\pi.
\end{aligned}$$

But we could also compute the area of a part of the sphere, say the band in the middle. Then we'd have

$$\begin{aligned}
SA &= \int_{-1/2}^{1/2} 2\pi\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}\,dx \\
&= 2\pi\int_{-1/2}^{1/2} \sqrt{1-x^2}\sqrt{\frac{1-x^2+x^2}{1-x^2}}\,dx \\
&= 2\pi\int_{-1/2}^{1/2} \sqrt{1-x^2}\sqrt{\frac{1}{1-x^2}}\,dx \\
&= 2\pi\int_{-1/2}^{1/2} 1\,dx = 2\pi.
\end{aligned}$$

So the middle half of the sphere has exactly half the surface area of the whole sphere!

Example 3.24. Let $f(x) = \sqrt[3]{3x}$. Take the portion of the graph where $0 \leq y \leq 2$ and rotate it around the y axis. What is the surface area? (See figure 3.9.)

This one will be easiser, for multiple reasons, to view as a function of y. So we have $y = \sqrt[3]{3x}$ and thus $x = y^3/3$. Then $x' = y^2$, and we have

$$\begin{aligned}
SA &= \int_0^2 \frac{2\pi y^3}{3}\sqrt{1+y^4}\,dy \\
&= \frac{2\pi}{3}\int_0^2 y^3\sqrt{1+y^4}\,dy \\
&= \frac{2\pi}{3}\frac{2}{12}(1+y^4)^{3/2}\Big|_0^2 \\
&= \frac{\pi}{9}\left(17^{3/2}-1\right) \approx 24.118.
\end{aligned}$$

And we can finish up with my favorite application/paradox, combining area, surface area, and improper integrals.

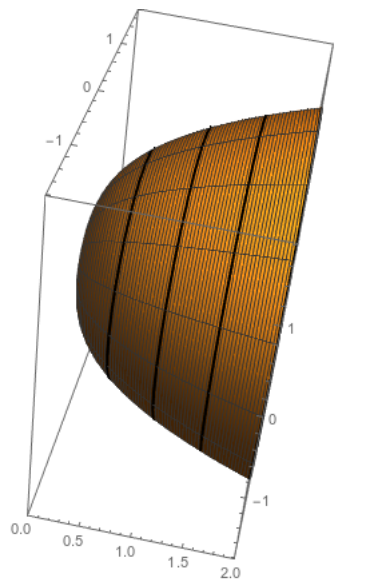


Figure 3.9: The graph of $y = \sqrt[3]{3x}$ rotated around the x-axis

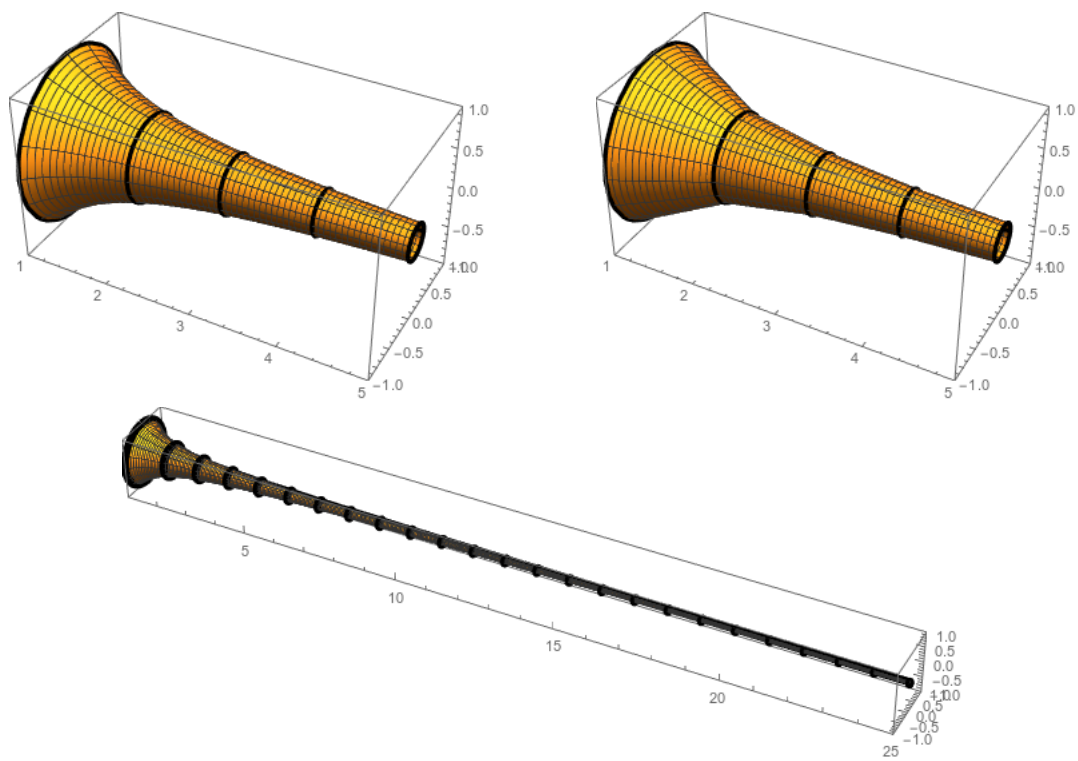


Figure 3.10: Gabriel's Trumpet

Example 3.25 (Gabriel's Trumpet/Infinite Paint Can)**.** Consider a trumpet-shaped container, given by taking the curve $y = 1/x$ and rotating around the x-axis, for $x \geq 1$. We're going to imagine this as a giant, oddly-shaped paint can. (See figure 3.10.)

We can work out the volume of this shape fairly easily, using cross-sections. If we take cross-sections perpendicular to the x-axis, each cross section is a circle of radius $1/x$. The area of this circle will be $\frac{\pi}{x^2}$ and thus the total volume will be

$$\begin{aligned}\int_1^\infty \frac{\pi}{x^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{\pi}{x^2}\,dx \\ &= \lim_{t\to\infty} \frac{-\pi}{x}\Big|_1^t = \frac{-\pi}{t} - \frac{-\pi}{1} = \pi.\end{aligned}$$

Thus the volume of our paint can is π; the can can hold π gallons of paint.

But now let's imagine painting the paint can. How much paint would we need to cover it? What's the surface area of the can?

We can do our surface area setup. We have $f(x) = 1/x$ so that $f'(x) = -1/x^2$. Then the surface area is

$$\begin{aligned}\int_1^\infty \frac{2\pi}{x}\sqrt{1+(-1/x^2)^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{2\pi}{x}\sqrt{1+1/x^4}\,dx \\ &\geq \lim_{t\to\infty} 2\pi \int_1^t \frac{1}{x}\,dx \\ &= 2\pi \lim_{x\to\infty} \ln|x|\Big|_1^t \\ &= \lim_{t\to\infty} 2\pi(\ln|t| - 1) = \infty.\end{aligned}$$

So the surface area of the paint can is infinite! You can fill the entire can with π gallons of paint, but it would take an infinite amount of paint to cover the interior of the paint can.

3.3 Differential Equations

There are two fundamentally different ways to think about what an integral is doing: as Riemann sums, or as antiderivatives. In calculus 1 and in the previous sections, our applications focused almost entirely on the Riemann sum aspect of things: we use integrals to split a question up into small pieces, approximating the answer on each piece, and then adding those approximations back together.

In this section we will look at the other side of the coin: how can we use *antiderivatives* to answer physical or practical questions? In this case, the questions will usually involve a derivative, so that the answer involves computing an antiderivative.

Definition 3.26. A *differential equation* is an equation that relates the derivatives of a function to the function itself.

Example 3.27.

- $y' = 2x$ is a differential equation; we can see that $y = x^2$ is a solution.

- $y' - 2y = 4\cos(t) - 8\sin(t)$ has a solution $y = 3e^{2t} + 4\sin(t)$ because
$$y' - 2y = 6e^{2t} + 4\cos(t) - \left(6e^{2t} + 8\sin(t)\right) = 4\cos(t) - 8\sin(t).$$
- $x^2y'' + xy' + x^2y = 0$ is *Bessel's differential equation of order 0* and has a solution called J_0 or *Bessel's Function of order 0*, as seen in figure 3.11. It is important in many applications, in particular studying heat flow.

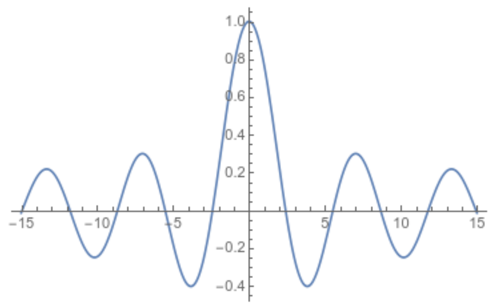


Figure 3.11: The graph of the Bessel function of order 0 J_0

Example 3.28. Confirm that $f(x) = x^2 + x + 1$ satisfies $2f(x) - xf'(x) = x + 2$.

We compute $f'(x) = 2x + 1$, so $2f(x) - xf'(x) = 2x^2 + 2x + 2 - (2x^2 + x) = x + 2$.

In a real way, a differential equations are the combination of high school algebra with precalculus and calculus. In grade school, we learned to do simple arithmetic, like being asked to compute $3 + 5$ and calculating 8. As we got to algebra, we were asked instead to *solve equations*. We would get formulas like $3 + x = 8$ and try to figure out what x is. This is the same sort of question but backwards—instead of computing with known numbers, we have to figure out which numbers will make the calculation work.

In pre-calculus and in calculus, we have done calculuations with functions. Plug a number into this function; graph this function; take the derivative of this function. But here we are being asked to solve equations whose answers are *functions*. The question is, which function satisfies the given relationship? And if we have a candidate answer, we can test it by plugging it into the differential equation and seeing if the computation we get is correct.

3.3.1 Proportional Growth

The simplest possible (non-trivial) differential equation is probably $p'(t) = kp(t)$. This tells us that the rate of change of something we're measuring is proportional to the current level of that thing.

This often comes up in the context of population growth. If we look at, say, a breeding population of rabbits, then the number of new rabbits born each year depends on the number of rabbits that are already alive: if we start with two rabbits, we won't end the year with two million. If each rabbit on average produces three new rabbits in a year, we might approximate the derivative by saying $\frac{dp}{dt} = 3p(t)$. That is, the change in the total population of rabbits is equal to three times the current number of rabbits.

In this case, if we start a year with 100 rabbits, then we have $p'(0) = 3p(0) = 300$ so we expect to get three hundred new rabbits, and end the year with 400. The next year we will get $p'(1) = 3p(1) = 1200$, so we get 1200 new rabbits and end the year with 1600 rabbits. The derivative is different each year, but the proportional growth rate is not.

Can we find a function that satisfies $p'(t) = 3p(t)$? And so far in this course, the answer is "not really". The trivial solution will still work, actually: if we start with zero rabbits, then we will always have zero rabbits, and it is true that $0' = 3 \cdot 0$. But if we want a non-trivial solution, none of the functions we've seen so far will work here. We will see that the solutions to this differential equation look like $p(t) = Ca^t$ for some constants C and a; a depends on the breeding rate, and C is the initial population of rabbits.

But this equation describes more than just rabbit population growth. Other cases where this equation appears include:

- Interest: if you are paying 8% interest per year, then your debt increases at a rate $d'(t) = .08d(t)$. This is the question Jakob Bernoulli was studying when he discovered the number e.

- Economic growth: the economy grows by 3% a year, so we have $p'(t) = .03p(t)$.

- Radioactive decay: some fraction of your sample of uranium will decay every year, so you have $u'(t) = ku(t)$. In this case k will be negative since your amount of uranium is decreasing.

- Heat transfer: the rate at which heat flows from a hot object to a cold object is proportional to the difference in temperature, so we have $T'(t) = kT(t)$.

3.3.2 Another Perspective on Compound Interest

Suppose you invest \$100 in a bank account paying 3% interest a year. Then after t years you will have $100 \cdot (1.03)^t$ dollars in the bank account. It's easy to compute how much money you'll have after t years. For instance, after three years you will have \$109 and after 20 years you will have \$180.

Often interest is "compounded" more often, meaning that you get some fraction of it every few months. Interest that is compounded quarterly—four times a year—pays you .75% of your current balance four times a year, so after t years you will have $100 \cdot (1.0075)^{4t}$ dollars. After three years you will still have \$109, and after 20 years you will have \$182. Note that your money has increased—slightly.

We can compound more often; in general, if your interest rate is r and you compound n times a year, then your total money after t years will be

$$M = M_0(1 + \frac{r}{n})^{nt},$$

where M_0 is the amount of money you started with.

In the real economy, transactions are constantly happening and the economy is (usually) constantly growing. Jacob Bernoulli asked what would happen if your interest *compounded continuously*—that is, what happens in the limit, as n goes to $+\infty$.

$$M(t) = \lim_{n \to +\infty} M_0 \left(1 + \frac{r}{n}\right)^{nt} = M_0 \left(\lim_{n \to +\infty} \left(1 + \frac{r}{n}\right)^{n/r} \right)^{rt} = M_0 e^{rt}.$$

And this was the context in which Bernoulli found the definition for e we gave in section 1.2.1.

Remark 3.29. This setup justifies a famous rule of thumb in finance. If we want to double our money, we're solving the equation

$$\begin{aligned} 2M_0 &= M_0 e^{rt} \\ \ln(2) &= rt \\ t &= \frac{r}{\ln(2)} \approx \frac{r}{.7}. \end{aligned}$$

This gives us the useful rule of thumb that if your interest rate is r, it will take about $70/r$ years to double your investment.

3.3.3 Force and Hooke's Law

For another common example, consider the phrase "acceleration is proportional to force." Recall that acceleration is the second derivative of position. If force is itself a function of position, this translates to a differential equation, relating $f''(x)$ to $f(x)$.

Hooke's law tells us that the force a spring exerts is proportional to the displacement of the spring; that is, for any given spring there is some constant k such that $F(t) = -kx(t)$, where $x(t)$ is the function that takes in the time and outputs the x coordinate of the weight on the spring. Since $F(t) = ma(t) = mx''(t)$, this gives us the differential equation $mx''(t) = -kx(t)$ or

$$x''(t) = -\frac{k}{m}x(t).$$

For simplicity let's assume $k = m$ so we have $x''(t) = -x(t)$.

Can we find a solution for this? We can start with the really silly or "trivial" solution. If the spring starts at a neutral position, it will never move, so we'd expect $x(t)$ to be constantly 0. And indeed this solution works: $0'' = 0 = -0$, so the funcion $x(t) = 0$ is a solution to this differential equation.

Can we find a solution that involves any motion at all? We're looking for a function where $x''(t) = -x(t)$. And we actually know two of these: $x(t) = \sin(t)$ and $x(t) = \cos(t)$ both satisfy this differential equation. And this is why the equation for "simple harmonic motion" is built up out of sin and cos functions.

There are many different solutions we can use; for example, $3\sin(t) + 5\cos(t) = 17$ is a solution to this differential equation. It's easy to see that if a and b are any constants, then $x(t) = a\sin(t) + b\cos(t)$ is a solution to this differential equation. It's much less obvious, but true, that any solution to the Hooke's Law equation must have this form; even the trivial solution is given by $x(t) = 0\sin(t) + 0\cos(t)$. Thus we say the *general form of the solution* is

$$x(t) = a\sin(t) + b\cos(t).$$

To pick out the specific solution we need to know some "initial conditions", which tell us what state the weight starts in. But if we know the starting position and starting velocity of the weight, we can determine a and b and thus get an exact formula for $x(t)$.

Example 3.30. Suppose we have a weight on a spring satisfying the differential equation $x''(t) = -x(t)$. Further, suppose we know that $x(0) = 3$ and $x'(0) = -1$: that is, at time 0 the weight is three units to the right of neutral, and is moving to the left with speed 1.

We know that $x(t) = a\sin(t) + b\cos(t)$ for some numbers a and b, and thus we have the equations

$$\begin{aligned} 3 &= x(0) = a\sin(0) + b\cos(0) = b \\ -1 &= x'(0) = a\cos(0) - b\sin(0) = a \end{aligned}$$

so our equation is in fact $x(t) = -\sin(t) + 3\cos(t)$, as shown in figure 3.16

Remark 3.31. Can you find a solution to $x''(t) = -4x(t)$?

Can you find a solution to $x''(t) = 4x(t)$?

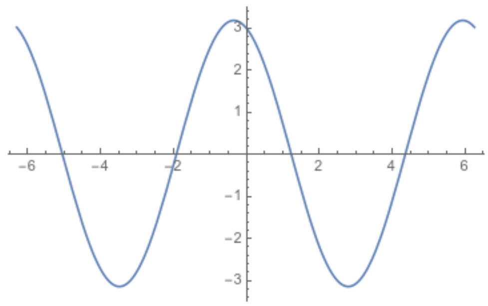


Figure 3.12: A graph of the solution to our differential equation. Notice how when the input is zero, the output is equal to three, but decreasing.

3.3.4 Evans price change model

Economists often use systems of differential equations to describe how the economy changes over time.

If there is a shortage of some good, which means that more people want to buy than sell, the price will tend to increase so that fewer people want to buy, more people want to sell, and the market clears. But the price doesn't change immediately. The Evans model says that the price change is proportional to the size of the shortage: $\frac{dp}{dt} = k(D - S)$, where D is the quantity demanded and S is the quantity supplied. So if the shortage is bigger, the price will increase faster.

So far, this looks sort of like exponential growth. But it's importantly different, because the size of the shortage is not the same thing as the price! We need to ask how demand depends on price. A simple model says that $D(p) = a - bp$ and $S(p) = r + sp$, where a is the amount demanded when the price is zero and r is the (probably negative) amount supplied when the price is zero. Then $-b = \frac{dD}{dp}$ and $s = \frac{dS}{dp}$ are the elasticities of demand and supply.

Plugging this back into the original model gives

$$p'(t) = k\Big(a - bp(t) - r - sp(t)\Big) = k\Big((a - r) - (b + s)p(t)\Big).$$

From this we can see that the trivial solution where the price is zero doesn't actually work here. And that makes sense, because if the price is zero you expect more people to want to buy than to sell. We also notice that it doesn't matter what the demand or supply elasticities are individually; it only matters what their sum is. We can use this equation to estimate the way the price will change over time.

Example 3.32. Let's assume $k = 1$, and take $a = 80, r = -20, b = s = 5$. That is, we have

$$\begin{aligned} D(p) &= 80 - 5p \\ S(p) &= -20 + 5p \\ p'(t) &= D(p) - S(p) && = 100 - 10p(t) \end{aligned}$$

Then we can check that a solution is $p(t) = 10 + 5e^{-10t}$, in which case the price will evolve following the path in figure 3.13.

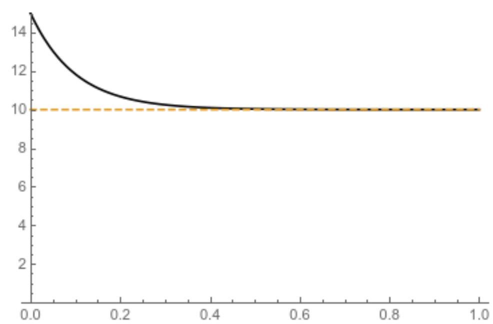


Figure 3.13: A graph of the function $p(t) = 10 + 5e^{-10t}$ solving the Evans Price Equation in example 3.32.

There is a rich and powerful theory for solving differential equations. We won't really be studying it in depth in this course; we would need a bunch of tools we don't have yet, including Taylor series (from section 5) and linear algebra (Math 2184). But we can learn about a few basic ideas.

3.3.5 Initial Value Problems

The harder part of solving differential equations is finding the general form of a solution. A given differential equation will usually have infinitely many solutions, as we saw in section 3.3.3 with the solutions $a\sin(t) + b\cos(t)$ to the equation $x''(t) = -x(t)$. This tells us the general shape of the solution, but doesn't give us an actual solution.

As we discussed, the specific solution depends on where things start. On the Hooke's Law fall system, if your fall starts at neutral then it will never move; if it starts extremely displaced then it will oscillate wildly. So to know the position over time we need to know where the system starts, known as the *initial conditions*. Finding a specific solution, given some specific conditions, is called an *initial value problem* or *boundary value problem*.

Example 3.33. Suppose we have a Hooke's Law system with $m = k$, so that we get the differential equation $x''(t) = -x(t)$. We said earlier that then $x(t) = a\sin(t) + b\cos(t)$ for some constants a and b.

Suppose now we start with the weight stationary and displaced by 1 meter. Since this is the starting conditions, this is at time 0, so this means that $x(0) = 1$ and $x'(0) = 0$. Now we have enough information to figure out a and b and find a specific solution to describe the path of our fall.

Since $x(0) = 1$ we know that

$$1 = a\sin(0) + b\cos(0) = b,$$

and since $x'(0) = 0$ we know that

$$0 = a\cos(0) - b\sin(0) = a$$

so we have $a = 0, b = 1$, and $x(t) = \cos(t)$, as graphed in

And as we think about it, this answer makes some sense: there's no reason for the fall to ever displace further than one meter, and so that's exactly what we see here.

Sometimes instead of an initial value problem we have a boundary value problem. In a boundary value problem you get position values at different times, rather than position and velocity at the same time.

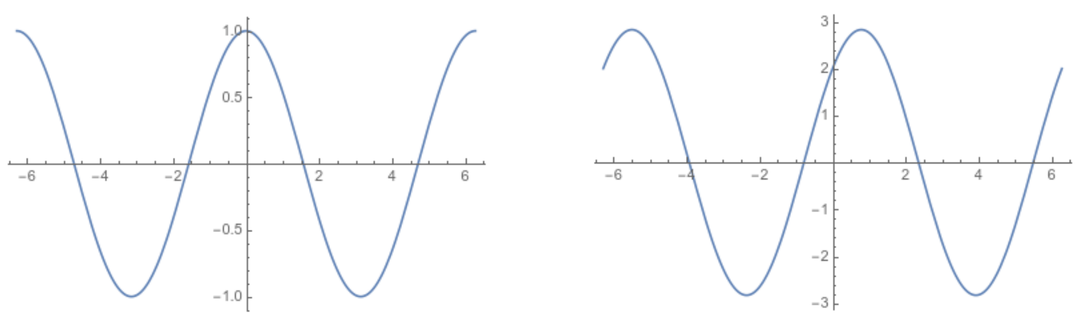


Figure 3.14: Left: the solution to example 3.33. Right: the solution to example 3.34

Example 3.34. Suppose we have a Hooke's Law setup with $m = k$, so $x(t) = a\sin(t) +$

$b\cos(t)$. Suppose we know that $x(0) = 2$ and $x(\pi/4) = \sqrt{8}$. Then we know that

$$\begin{aligned}
a\sin(0) + b\cos(0) &= 2 \\
b &= 2 \\
a\sin(\pi/4) + b\cos(\pi/4) &= \sqrt{8} \\
a\sqrt{2}/2 + 2 \cdot \sqrt{2}/2 &= \sqrt{8} \\
a/2 + 1 &= 2 \\
a &= 2.
\end{aligned}$$

Thus we have that $x(t) = 2\sin(t) + 2\cos(t)$.

Notice that in either of these cases, we need to take only two measurements to know exactly what happens at every possible time. This is because our differential equation, coming from a physical law, severely constrains what our answers can possibly look like; we only need a bit more information to have it nailed down precisely, one measurement for each constant.

Of course, in the real world, measurements come with errors so we need to take more than two. But we can get a lot of information from our differential equation telling us what sort of relationships to look for.

Example 3.35. Suppose $f(x) = ax^2 + bx + c$ is a polynomial satisfying some differential equation, and we have $f(0) = 0, f'(0) = 1, f''(0) = 2$. What can we say about $f(x)$?

We see that $f(0) = c = 0$, $f'(x) = 2ax + b$ so $f'(0) = b = 1$, and $f''(x) = 2a$ so $f''(0) = 2a = 2$. Thus $a = b = 1$ and $c = 0$, so $f(x) = x^2 + x$.

Example 3.36. Suppose $g(x) = ax^2 + bx + c$ is a polynomial satisfying some differential equation, with $g(1) = 2, g'(2) = 3, g''(3) = 4$. What can we say about g?

We have $g(1) = a + b + c$. $g'(x) = 2ax + b$ so $g'(2) = 4a + b = 3$, and $g''(x) = 2a$ so $g''(3) = 2a = 4$. Thus we have $a = 2$. Going back to g' we see that $8 + b = 3$ so $b = -5$. Then plugging into g we have $2 - 5 + c = 2$ so $c = 5$. Thus $g(x) = x^2 - 5x + 5$.

3.4 Separable differential equations

Even ordinary differential equations are hard to solve; in general solving them involves using power series (see section 5) and Fourier series (which we won't really discuss in this course). But there is one fairly easy type of equation that we do have the tools to solve.

Definition 3.37. A *separable differential equation* is a differential equation that can be written
$$\frac{dy}{dx} = g(x)f(y)$$
for some functions g and f.

We call these separable because we can separate the variables, by putting all the ys on one side and all the xs on the other. Heuristically, we divide by $f(y)$ and "multiply by dx": this gives us
$$\frac{dy}{f(y)} = g(x)\,dx$$
and we can now integrate both sides. We can justify this via the chain rule: if
$$\int \frac{dy}{f(y)} = \int g(x)\,dx$$
then
$$\begin{aligned}
\frac{d}{dx}\left(\int \frac{dy}{f(y)}\right) &= \frac{d}{dx}\left(\int g(x)\,dx\right)\\
\frac{dy}{dx}\cdot\frac{d}{dy}\left(\int \frac{dy}{f(y)}\right) &= \frac{d}{dx}\left(\int g(x)\,dx\right)\\
\frac{dy}{dx}\cdot\frac{1}{f(y)} &= g(x).
\end{aligned}$$

Alternatively, we can justify it with u-substitution. If $\frac{dy}{dx} = g(x)f(y)$, then
$$\begin{aligned}
\frac{1}{f(y)}\frac{dy}{dx} &= g(x)\\
\int \frac{1}{f(y)}\frac{dy}{dx}\,dx &= \int g(x)\,dx\\
\int \frac{1}{f(u)}\,du &= \int g(x)\,dx.
\end{aligned}$$

Example 3.38. Solve $y' = x/y$ for the initial value $y(0) = 2$.

We have
$$\begin{aligned}
\frac{dy}{dx} &= \frac{x}{y}\\
\int y\,dy &= \int x\,dx\\
\frac{y^2}{2} &= \frac{x^2}{2} + C\\
y &= \pm\sqrt{x^2 + 2C}
\end{aligned}$$

This gives us our general solution; now we just need to find the specific solution.

our initial condition is $y(0) = 2$, so we have $\pm\sqrt{0+2C} = 2$ and thus $C = 2$. This gives $y = \pm\sqrt{x^2+4}$. Since our square root must be positive, we get a specific solution $y = \sqrt{x^2+4}$.

(Note that if our initial condition were negative, say $y(0) = -2$, then we'd have a negative square root instead, and $y = -\sqrt{x^2+4}$.)

Remark 3.39. If we want, we could have replaced the $2C$ with a C without losing anything. Alternatively we could set $K = 2C$ to get the same effect but make sure we don't confuse ourselves.

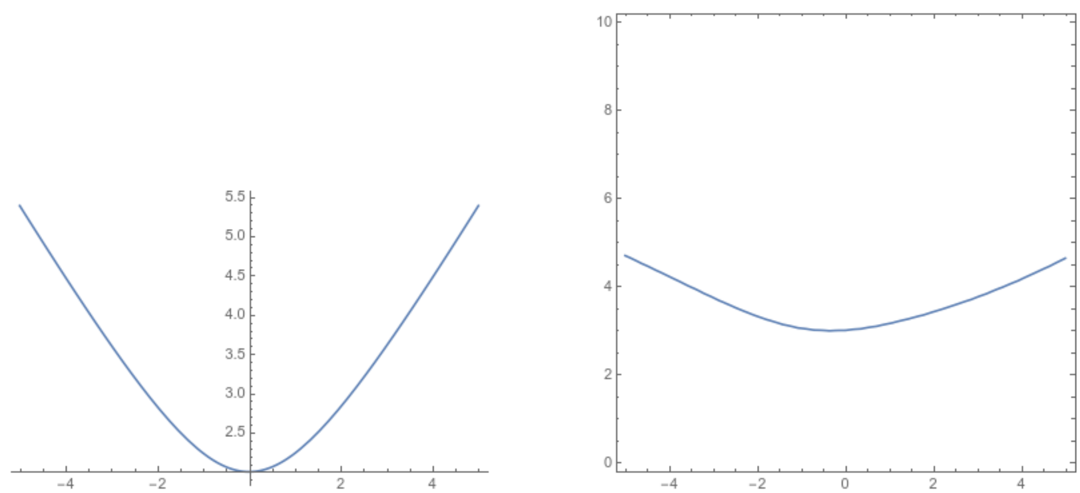


Figure 3.15: Left: the solution to example 3.38. Right: the solution to example 3.40

Example 3.40. Solve $y' = \frac{3x+\cos x}{2y+y^2}$ for the initial condition $y(0) = 3$.

We have

$$\frac{dy}{dx} = \frac{3x+\cos x}{2y+y^2}$$
$$\int 2y + y^2\,dy = \int 3x + \cos x\,dx$$
$$y^2 + \frac{y^3}{3} = \frac{3x^2}{2} + \sin x + C.$$

This gives y as a function of x implicitly, but there's no immediately obvious way to write it explicitly as a function of x. We can, however, work out the constant; we have $3^2 + 3^3/3 = 0 + 0 + C$ and thus $C = 18$. So our solution is

$$y^2 + \frac{y^3}{3} = \frac{3x^2}{2} + \sin x + 18.$$

Example 3.41. Find the general solution to $(y^2 + xy^2)y' = 1$.

We have

$$\int y^2 dy = \int \frac{dx}{1+x}$$
$$\frac{y^3}{3} = \ln|1+x| + C$$
$$y = \sqrt[3]{3\ln|1+x| + 3C}.$$

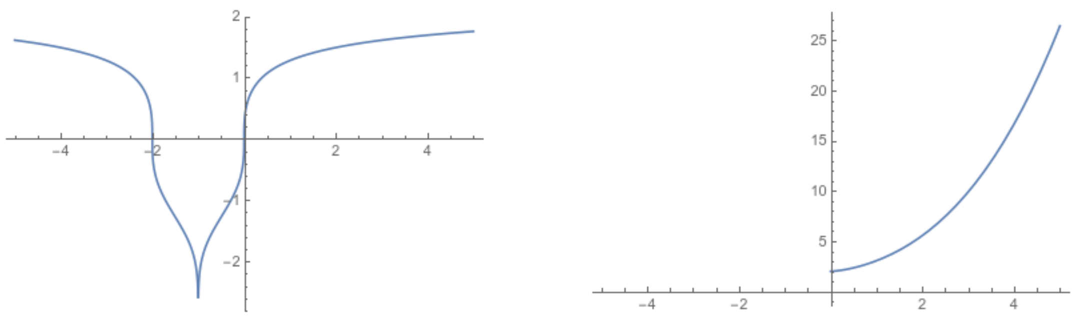


Figure 3.16: Left: solution to example 3.41 for $C = 0$. Right: the solution to example 3.42.

Example 3.42. Find the specific solution to $P' = \sqrt{Pt}$ with $P(0) = 2$.

We have

$$\int \frac{dP}{\sqrt{P}} = \int \sqrt{t}\, dt$$
$$2\sqrt{P} = \frac{2}{3}t^{3/2} + C$$
$$P = \left(t^{3/2}/3 + C/2\right)^2$$

and since $P(1) = 2$ we have $2 = (0 + C/2)^2$ and thus $C = 2\sqrt{2}$. So the specific solution is

$$P = \left(\frac{t^{3/2}}{3} + \sqrt{2}\right)^2.$$

3.5 Some common separable differential equations

3.5.1 Mixing Problems

In section 3.3.1 we looked at proportional or exponential growth. A slightly more complicated variant on exponential growth occurs when there is constant growth and exponential decay at the same time (or vice versa).

Example 3.43. Suppose we have a tank containing 10 kg of salt dissolved in 1000 L of water. We can pump in a brine solution of .02 kg of salt per liter of water at 10L per minute, while ten L of solution drains out of the tank each minute. How much salt is in the tank after twenty minutes?

Let y be the amount of salt in the tank. Then we have $y(0) = 10$, and we have $y' = .2 - \frac{y}{100}$, since each minute the tank is gaining .1 kg of salt and losing a hundredth of its salt content.

We can easily rewrite this as

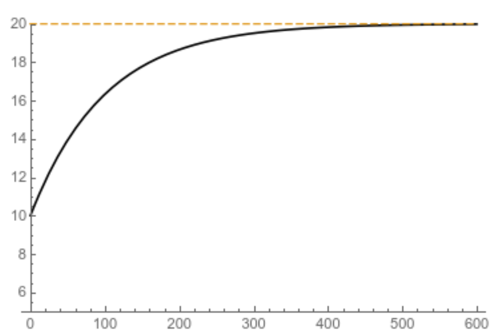
$$\begin{aligned} \frac{dy}{dt} &= \frac{20-y}{100} \\ \int \frac{dy}{20-y} &= \int \frac{dt}{100} \\ \ln|20-y| \cdot (-1) &= \frac{t}{100} + C \end{aligned}$$

Since $y(0) = 10$ so we have $-\ln(10) = C$, so we get

$$\begin{aligned} -\ln|20-y| &= \frac{t}{100} - \ln(10) \\ |20-y| &= 10e^{-t/100} \\ 20-y &= 10e^{-t/100} \\ y &= 20 - 10e^{-t/100}. \end{aligned}$$

Thus after 20 minutes we have

$$y = 20 - 10e^{-1/5} \approx 11.81kg.$$



You might recognize this formula: this is mathematically the same as the Evans price change model we saw in section 3.3.4. However, the "physics" giving rise to the equation here is somewhat different.

3.5.2 Logistic Growth

We saw that in a simple model of population growth, the population will grow exponentially. But this model assumes that there are no real resource constraints; the population will keep growing larger and larger as time goes on. In reality, they will eventually run out of space or food or some other resource.

A simple but important model for this is the model of *logistic growth*. Let M be the *carrying capacity*, i.e. the maximum population. Then when our population is small, we want growth roughly proportional to the size of our population, as before. But as the population gets closer to M the rate of growth gets closer to 0; a simple equation that captures this is the *logistic differential equation* first developed by Pierre-François Verhulst in the 1840s:

$$\frac{dy}{dt} = ky(M-y).$$

We can see that if M is much bigger than y, this is approximately $\frac{dy}{dt} = kMy$ and thus is exponential growth. But if y is very close to M, we have $\frac{dy}{dt} \approx ky \cdot 0$.

The equation is separable, so we can write

$$\begin{aligned} \int k\,dt &= \int \frac{dy}{y(M-y)} \\ &= \int \frac{1}{M}\left(\frac{1}{y} - \frac{1}{M-y}\right) dy \\ &= \frac{1}{M}\left(\int \frac{dy}{y} - \frac{dy}{M-y}\right) \\ kt + C &= \frac{1}{M}\left(\ln|y| - \ln|M-y|\right). \end{aligned}$$

Since $0 < y < M$ both y and $M-y$ are positive, so this gives

$$M(kt+C) = \ln\left(\frac{y}{M-y}\right)$$

and thus

$$\frac{y}{M-y} = Ae^{Mkt}.$$

Given an initial condition $y(0) = y_0$ we have

$$\frac{y}{M-y} = \frac{y_0}{M-y_0}e^{Mkt}$$

and solving for y to write y as a function of t gives us

$$\begin{aligned}
y &= (M-y)\frac{y_0}{M-y_0}e^{Mkt} \\
&= \frac{My_0}{M-y_0}e^{Mkt} - \frac{yy_0}{M-y_0}e^{Mkt} \\
y\left(1+\frac{y_0}{M-y_0}e^{Mkt}\right) &= \frac{My_0}{M-y_0}e^{Mkt} \\
y &= \frac{\frac{My_0}{M-y_0}e^{Mkt}}{1+\frac{y_0}{M-y_0}e^{Mkt}} \\
&= \frac{My_0e^{Mkt}}{M-y_0+y_0e^{Mkt}} \\
&= \frac{My_0}{(M-y_0)e^{-Mkt}+y_0}.
\end{aligned}$$

We can see that, as expected, as $t \to +\infty$ we have $y \to M$.

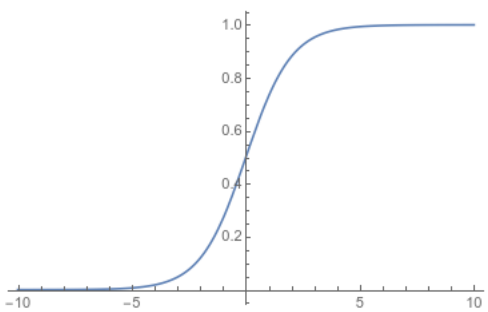


Figure 3.17: The characteristic S-curve of logistic growth. Here we take $M = 1, k = 1, y_0 = .5$.

Example 3.44 (Global Population)**.** The total population of the world was 3 billion people in 1960, and 4 billion in about 1975. We can try to model this with a simple exponential growth curve: Setting $t = 0$ to be 1960 and fitting this to our model, we have: $Ce^{k0} = 3$ and $Ce^{15k} = 4$. Thus we must have $C = 3$, and then $e^{15k} = 4/3$ implies that $15k = \ln(4/3)$ and so $k = \ln(4/3)/15$. If we want to estimate global population in 2020, this gives us

$$P(60) = 3 \cdot e^{60 \cdot \ln(4/3)/15} = 3 \cdot e^{4\ln(4/3)} = 3 \cdot (4/3)^4 \approx 9.48.$$

(Actual estimates put it at 7.8 billion, because population growth has been leveling off).

Now let's use our model to estimate when the population will reach 12 billion. We want

$$\begin{aligned}
12 &= 3 \cdot e^{t\ln(4/3)/15} \\
4 &= \left(\frac{4}{3}\right)^{t/15} \\
\log_{4/3} 4 &= t/15 \\
15\frac{\ln(4)}{\ln(4/3)} &= t \\
72.3 &\approx t
\end{aligned}$$

So our model predicts that the world's population will reach 12 billion in about 2032.

Now let's use a more sophisticated logistic growth model. Suppose

$$\frac{dP}{dt} = kP(M-P)$$

and thus

$$P = \frac{MP_0}{(M-P_0)e^{-Mkt}+P_0}.$$

We know that $P_0 = P(0) = 3$ if 0 corresponds to 1960. We also have $P(15) = 4$ and $P(60) = 7.8$. We need to solve for M and for k. Using the fact that $P(15) = 4$ we get have

$$\begin{aligned}
4 &= \frac{M \cdot 3}{(M-3)e^{-Mk\cdot 15}+3} \\
3M &= (4M-12)e^{-15Mk} + 12 \\
e^{-15Mk} &= \frac{3M-12}{4M-12} \\
-15Mk &= \ln\left|\frac{3M-12}{4M-12}\right| \\
k &= \frac{-1}{15M}\ln\left|\frac{3M-12}{4M-12}\right|.
\end{aligned}$$

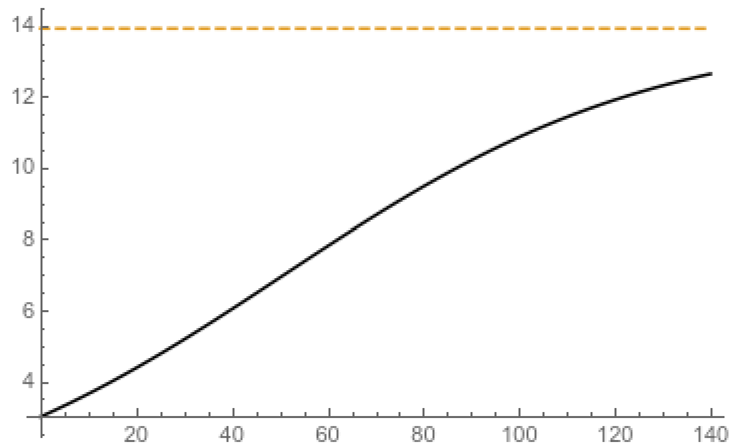
We now need to use the fact that $P(60) = 7.8$ to find actual values for k and M. Solving by hand would be terribly annoying, but I asked Mathematica and it told me that if $P(60) = 7.8$ then $M \approx 13.9$ and $k \approx .0018$ Plugging this in gives us

$$P(t) = \frac{39.6}{10.2e^{-.026t}+3}$$

which tells us $P(0) = 3, P(15) = 4, P(54) = 7.2$, and $P(60) = 7.7$ in line with current projections. Using this equation we can ask again when population will reach 12 billion; we see this happens when $t = 122$, or in about 2082.

The UN currently projects a population of about 11.2 billion in 2100, which is a bit less than we would predict. Our logistic model is much better than the exponential model, but not quite as good as you can do with professional demographers.

Figures 3.18 and 3.19 compre graphs of our model with the UN data, provided by Our World in Data. If you want to learn more about this topic, check out the Future Population Growth page at `https://ourworldindata.org/future-population-growth`.



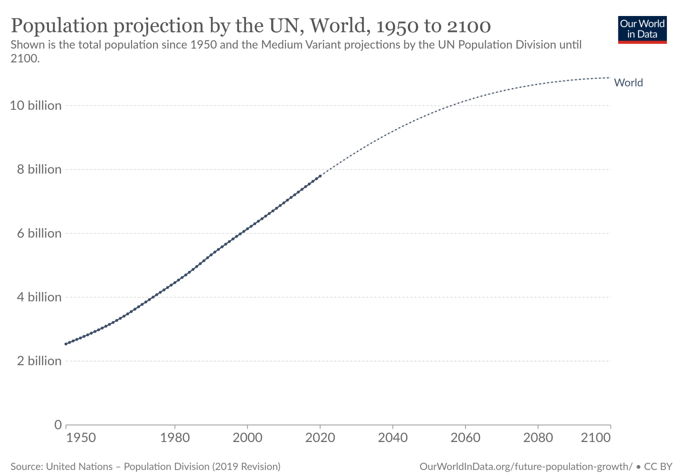


Figure 3.18: Top: our population projection through 2100. Bottom: the UN's projection.

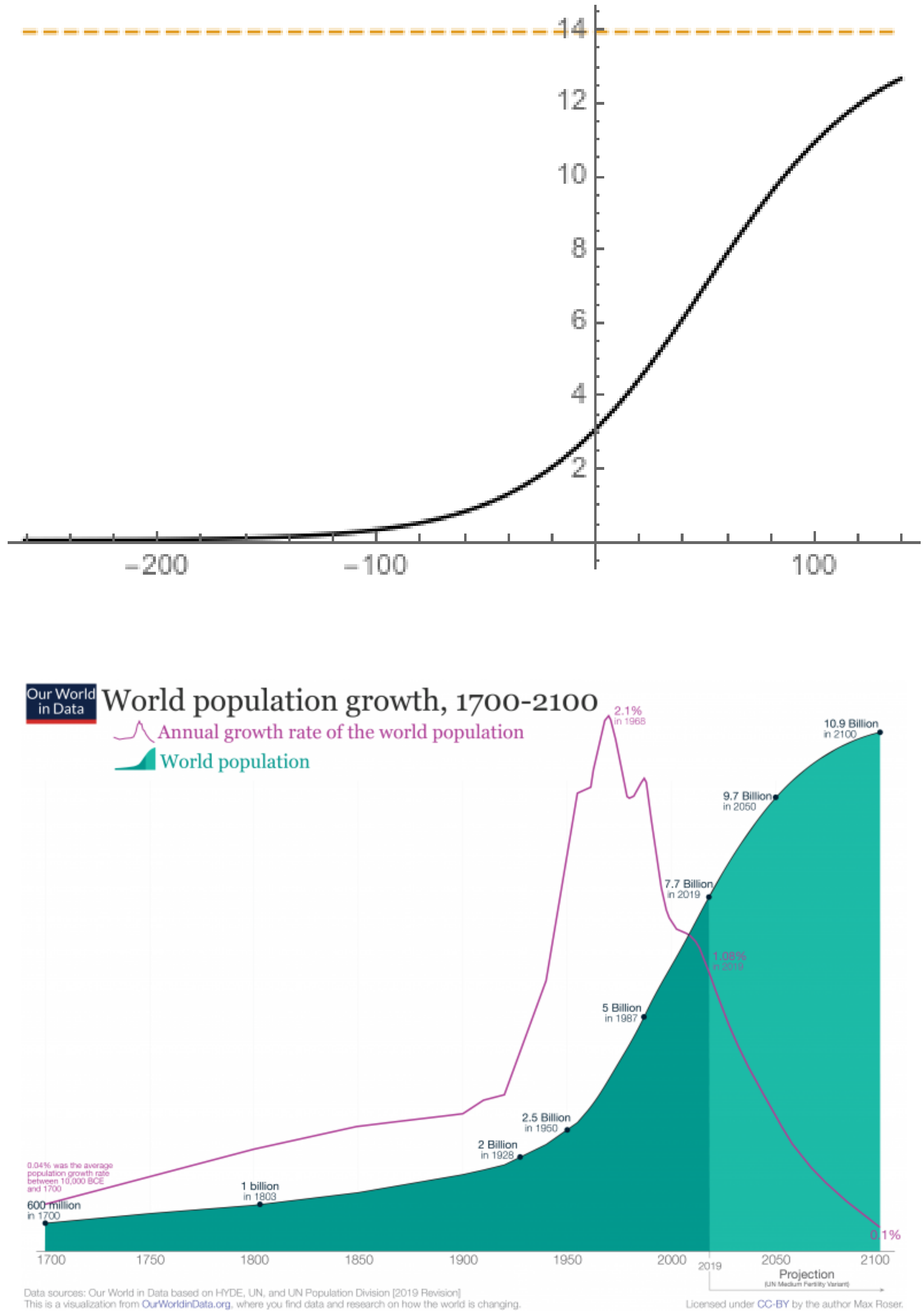


Figure 3.19: Top: our estimate of the population from 1700 through 2100. Bottom: Our World in Data's estimate (with graph of the estimated derivative).

4 Sequences and Series

In this section we're going to introduce a completely new set of ideas. Well, sort of.

So far in calculus we've studied *continuous* things: functions, derivatives, and integrals. What these all have in common is that you can chop them into pieces as small as you want. In this section we'll see what happens if we study *continuous* things, that genuinely have a smallest possible size.

Conceptually, thinking about discrete things is easier: there are fewer things that can happen, so everything is conceptually simpler. And in a real sense, the world as we perceive it is basically discrete, since you're never going to make infinitely many measurements.

So then… why did we start with the continuous version? It's conceptually harder and more artificial, but it is far, far easier to do *computations* in the continuous realm. So the way we mostly solve real problems is to find a way to pretend our discrete question is really continuous, and then solve the continuous question, and hope we can use that to answer our original discrete question. We'll see that throughout this section as well.

4.1 Sequences

Definition 4.1. A *sequence* of real numbers is a (usually infinite) ordered list of real numbers. We write $(a_n)_{n=1}^{\infty}$ for the sequence

$$(a_1, a_2, a_3, \dots)$$

where each a_n is a real number.

We can think of a sequence as the discrete equivalent of a function. In particular, a sequence is a function from the natural numbers to the real numbers, where $f(n)$ is the nth element of the sequence. Thus it's a function that only allows integer inputs, unlike continuous functions that allow any real number as an input.

Example 4.2. A few examples of sequences. Some of these will look familiar:

(a) $(1, 1, 1, 1, 1, \dots)$

(b) $(1, 2, 3, 4, \dots)$

(c) $(2^{10}, 17, \sqrt[5814]{3^{11} - 1}, 1, 1, 1, \dots)$

(d) $(1, 1, 2, 3, 5, 8, 13, \dots)$

(e) $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$

(f) $(3, 3.1, 3.14, 3.141, 3.1415, \dots)$

(g) $(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots)$

(h) $(1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1, \dots)$

In most of these sequences the pattern is pretty obvious. In sequence (a) we have $a_n = 1$. In sequence (b) we have $a_n = n$ and in sequence (e) we have $a_n = 1/n$. Less obviously, in sequence (g) we have $\frac{n}{n+1}$ and in (h) we have $\cos(n\pi/6)$.

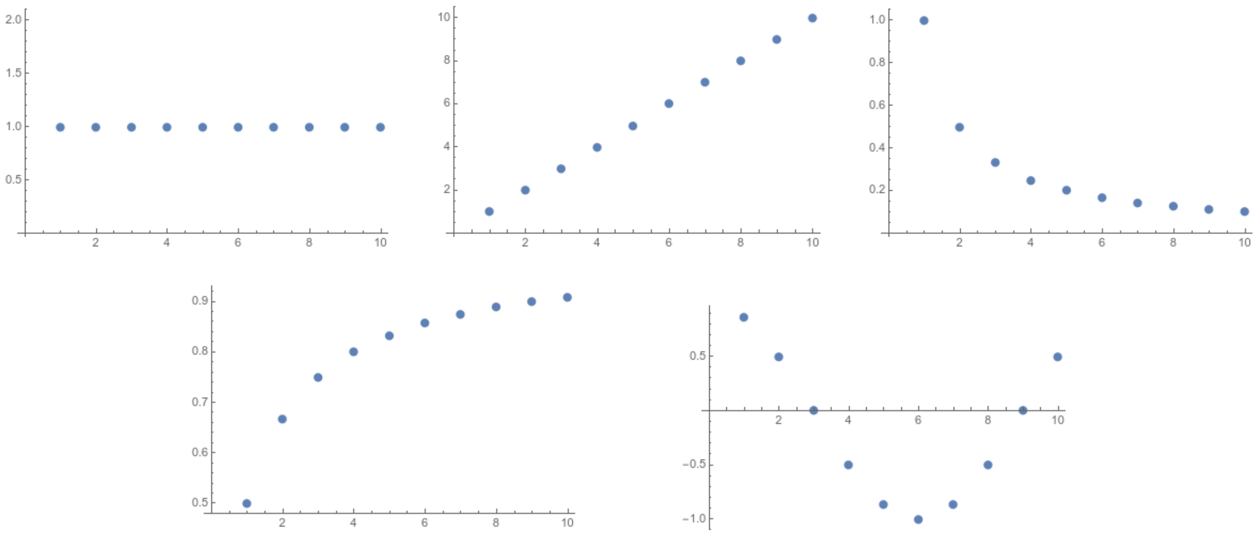


Figure 4.1: The graphs of (a), (b), (e), (g), (h)

However, not all sequences have nice descriptions like this. Sequence (d) is the *fibonacci sequence*, which is defined "inductively" or "recursively" by $f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. (This sequence was originally defined to work on problems about rabbit-breeding; it appears often in nature).

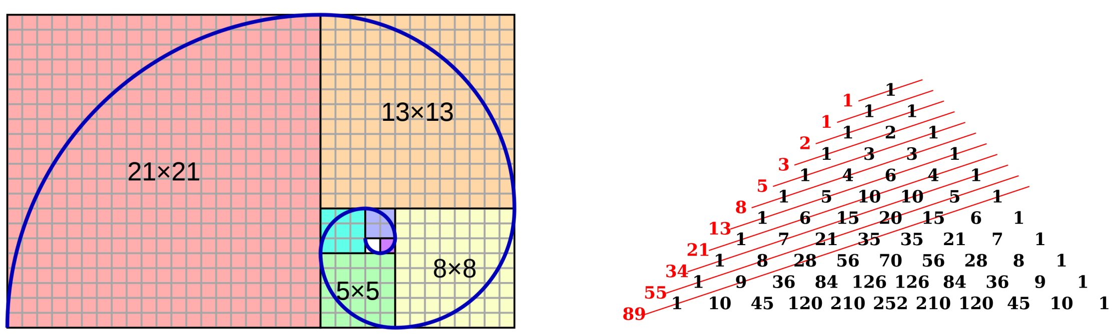


Figure 4.2: The Fibonacci sequence is related to the Golden Ratio and to Pascal's Triangle. Left: Jahobr, CC0; Right: RDBury, CC BY-SA 3.0, both via Wikimedia Commons

Even worse are sequences like (c) which show no particular pattern at all; these are still sequences.

Example 4.3. What is the general form of the sequence $(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots)$? We see that $a_n = \frac{1}{n^2}$.

What about $(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots)$? $a_n = \frac{1}{2^n}$.

We can see that some of these sequences look like they're "going somewhere"—in fact, sequence (a) is already there! But sequences (e), (f), and (g) all seem to be getting closer and closer to some value.

When the terms of a sequence are getting closer and closer to some value, we say that it has a *limit*. In particular, we say the sequence (a_n) has a limit L in the real numbers if we can make the numbers a_n get as close to L as we want just by taking n to be sufficiently big.

Example 4.4. The sequence $(1/n^2)$ has a limit of 0.

The sequence $1/2^n$ also has a limit of 0.

Just like with functions, we can use ϵ to make this definition more rigorous:

Definition 4.5. Let (a_n) be a sequence of real numbers. We say that (a_n) has a limit L, and write $\lim_{n\to+\infty} a_n = L$, if, for every real number $\epsilon > 0$, there is a natural number N such that, whenever $n \geq N$, $|a_n - L| < \epsilon$.

If a sequence has a limit in the real numbers we say the sequence *converges*. Otherwise we say the sequence *diverges*, and the limit *does not exist*.

Example 4.6. Prove that $\lim_{n\to+\infty} \frac{1}{n} = 0$.

Fix some $\epsilon > 0$. Then let $N > 1/\epsilon$. If $n \geq N$ then $\frac{1}{n} \leq \frac{1}{N} < \epsilon$, and thus $|a_n - 0| < \epsilon$. So by definition, $\lim_{n\to+\infty} \frac{1}{n} = 0$.

Example 4.7. Prove that $\lim_{n\to+\infty}(-1)^n$ does not exist.

Heuristically, we notice that this sequence "bounces around"; it doesn't get closer to just one value. Informally, the sequence has two different values it reaches infinitely often, so it doesn't have one single limit. But we can also make this rigorous with an $\epsilon - N$ argument:

For a limit to exist, a certain statement needs to be true for any positive real number. So to prove that a limit does not exist we just need to find *one* real number for which the statement is false.

So let $\epsilon = 1$, and suppose a limit L exists. Then we can find a N such that if $n \geq N$, then $|(-1)^n - L| < 1$. In particular, we can find both even n and odd n, and so it must be the case that $|1 - L| < 1$ and $|-1 - L| < 1$. But there is no number L that makes this true. So no limit exists.

Computing limits in this way is important, and a good exercise, but a bit painful. And just like with functions, we have limit laws that make the process much easier.

Proposition 4.8. *If (a_n) and (b_n) are convergent sequences and c is a constant, then*

- $\lim\limits_{n\to+\infty} a_n \pm b_n = \lim\limits_{n\to+\infty} a_n \pm \lim\limits_{n\to+\infty} b_n$
- $\lim_{n\to+\infty} c = c$
- $\lim_{n\to+\infty} ca_n = c\lim_{n\to+\infty} a_n$
- $\lim_{n\to+\infty} a_n b_n = \lim_{n\to+\infty} a_n \lim_{n\to+\infty} b_n$
- $\lim_{n\to+\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to+\infty} a_n}{\lim_{n\to+\infty} b_n}$ *if* $\lim_{n\to+\infty} b_n \neq 0.$
- $\lim_{n\to+\infty} a_n^p = (\lim_{n\to+\infty} a_n)^p$ *if* $p, a_n > 0.$

Example 4.9. What is $\lim_{n\to+\infty} \frac{n+1}{n}$?

We can write

$$\begin{aligned}\lim_{n\to+\infty} \frac{n+1}{n} &= \lim_{n\to+\infty} (1+\frac{1}{n}) \\ &= \lim_{n\to+\infty} 1 + \lim_{n\to+\infty} \frac{1}{n} = 1 + 0 = 1.\end{aligned}$$

Example 4.10 (recitation)**.** What is the limit of the sequence $\sqrt{n+1} - \sqrt{n}$?

Using a familiar trick from Calculus 1, we see

$$\begin{aligned}\sqrt{n+1} - \sqrt{n} &= \frac{(\sqrt{n+1}+\sqrt{n})(\sqrt{n+1}-\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} \\ &= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} \\ &= \frac{1}{\sqrt{n+1}+\sqrt{n}}.\end{aligned}$$

Thus

$$\lim_{n\to+\infty} \sqrt{n+1} - \sqrt{n} = \lim_{n\to+\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0.$$

That last step was arguably a bit fuzzy. There are a few ways to make it rigorous; one is to argue that our sequence "looks like" $\frac{1}{\sqrt{n}}$. In particular, our sequence is smaller than $\frac{1}{\sqrt{n}}$ and $\frac{1}{\sqrt{n}} \to 0$, so our sequence should also get close to zero. We can make that precise with the Squeeze Theorem:

Theorem 4.11 (Squeeze Theorem)**.** *If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} c_n = L$ then $\lim_{n\to+\infty} b_n = L$.*

To continue the earlier example, we have

$$\lim_{n\to+\infty} \sqrt{n+1}-\sqrt{n} = \lim_{n\to+\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$$
$$0 \leq \frac{1}{\sqrt{n+1}+\sqrt{n}} \leq \frac{1}{\sqrt{n}}.$$

We know that $\lim_{n\to+\infty} 0 = 0$. And we can work out that

$$\begin{aligned}\lim_{n\to+\infty} \frac{1}{\sqrt{n}} &= \lim_{n\to+\infty} \left(\frac{1}{n}\right)^{1/2} \\ &= \left(\lim_{n\to+\infty} \frac{1}{n}\right)^{1/2} \\ &= 0^{1/2} = 0.\end{aligned}$$

Then by the squeeze theorem, $\lim_{n\to+\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$.

Example 4.12. What is $\lim_{n\to+\infty} \frac{\sin n}{n}$?

This is a classic use case for the Squeeze Theorem. We know that $-1 \leq \sin n \leq 1$ for any n. So $\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$ for any n. We know that $\lim_{n\to+\infty} \frac{1}{n} = 0$, and similarly $\lim_{n\to+\infty} \frac{-1}{n} = -\lim_{n\to+\infty} \frac{1}{n} = 0$. So by the squeeze theorem, $\lim_{n\to+\infty} \frac{\sin n}{n} = 0$.

But ultimately we can replace a lot of this with all the work we did with functions—since a sequence is secretly just a function anyway.

Theorem 4.13. *Suppose $f(x)$ is a function such that $f(n) = a_n$ for every natural number n, and $\lim_{x\to+\infty} f(x) = L$. Then $\lim_{n\to+\infty} a_n = L$.*

If $\lim_{n\to+\infty} a_n = L$ and f is continuous at L, then

$$\lim_{n\to+\infty} f(a_n) = f(L).$$

Example 4.14. What is $\lim_{n\to+\infty} \frac{n}{n+1}$? We see that if $f(x) = \frac{x}{x+1}$, then $f(n) = a_n$, so we can compute

$$\lim_{n\to+\infty} \frac{n}{n+1} = \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x/x}{(x+1)/x} = \lim_{x\to+\infty} \frac{1}{1+1/x} = 1.$$

Thus $\lim_{n\to+\infty} \frac{n}{n+1} = 1$.

Example 4.15. What is $\lim_{n\to+\infty} \frac{\ln n}{n}$?

We write $f(x) = \frac{\ln x}{x}$, and then $f(n) = a_n$. By L'Hôpital's rule, we have

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{1/x}{1} = \lim_{x\to+\infty} \frac{1}{x} = 0.$$

Thus we also have that $\lim_{n\to+\infty} \frac{\ln n}{n} = 0$.

However, this only works in one direction! If the function limit exists, then the sequence limit exists. But the converse is not true.

Example 4.16. What is $\lim_{n\to+\infty} \sin(n\pi)$?

Naively, we might argue this: Let $g(x) = \sin(x\pi)$. Then $\lim_{x\to+\infty} g(x)$ does not exist, since the function varies between -1 and 1 no matter how large we let x grow. Thus the limit does not exist.

However, our theorem only applies when $\lim_{x\to+\infty} g(x)$ exists; it tells us nothing if the limit of our function does not converge. In fact, for every n we have $\sin(n\pi) = 0$, and thus

$$\lim_{n\to+\infty} \sin(n\pi) = \lim_{n\to+\infty} 0 = 0.$$

But the real limitation is: not every sequence can be expressed reasonably as a function of the real numbers at all.

Definition 4.17. If n is a natural number, we define n *factorial*, written $n!$, to be

$$n! = n \cdot (n-1) \dots 2 \cdot 1.$$

This is the product of all positive integers less than or equal to n.

These will come up a lot in the remainder of this course.

Example 4.18. What is $\lim_{n\to+\infty} \frac{n!}{n^n}$?

We calculate that

$$a_n = \frac{n!}{n^n} = \frac{n(n-1)(n-2)\dots(2)(1)}{n \cdot n \cdot n \dots n \cdot n} = \frac{1}{n} \cdot \frac{n(n-1)(n-2)\dots(2)}{n^{n-1}}.$$

It's clear that the large fraction is between 0 and 1 since the numerator is positive, but smaller than the denominator. Thus we have $0 \leq a_n \leq \frac{1}{n}$, and $\lim_{n\to+\infty} 0 = \lim_{n\to+\infty} \frac{1}{n} = 0$. By the squeeze theorem, $\lim_{n\to+\infty} a_n = 0$.

And just like with functions, we sometimes have sequences wtih infinite limits.

Example 4.19. $\lim_{n\to+\infty} n = +\infty$.

$\lim_{n\to+\infty} -n^2 = -\infty$.

4.1.1 Completeness

There's one important note I want to make here about the way sequences work, and the importance of the real numbers.

We would like to say that every sequence either goes to infinity or has a (finite) limit. Unfortunately, this isn't the case, because a sequence can bounce up and down without ever settling on one value (remember $(-1)^n$). But if a sequence doesn't "bounce around" then we know it must either have a limit or go to infinity.

Definition 4.20. A sequence is *(monotonically) increasing* if $a_{n+1} \geq a_n$ for all n. A sequence is *(monotonically) decreasing* if $a_{n+1} \leq a_n$ for all n. In either case we say that such a sequence is *monotonic.*

A sequence is *bounded above* if there is an A such that $a_n \leq A$ for all n. A sequence is *bounded below* if there is an A such that $a_n \geq A$ for all n. A sequence that is bounded above and bounded below is *bounded.*

A monotone sequence doesn't bounce around; a bounded sequence doesn't go to infinity. In the real numbers, a sequence with both of these properties must have a limit.

Fact 4.21. *Every increasing sequence of real numbers that is bounded above converges to some real number. Every decreasing sequence of real numbers that is bounded below converges to some real number. In particular, every bounded monotonic sequence is convergent.*

Remark 4.22. The idea here is that every sequence that "should" have a finite limit does. If the terms get closer to each other, there is some limit they approach.

Example 4.23. $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$.

If $0 \leq x \leq 2$ then $x \leq \sqrt{2x} \leq 2$. Thus since the first element is between 0 and 2, the sequence is increasing, and every element is ≤ 2, so the sequence is bounded above by 2. Thus it must converge.

Can we see what it must converge to? If we look at the sequence $a_n^2/2$ we have $1, \sqrt{2}, \sqrt{2\sqrt{2}}, \ldots$ and get the same sequence again, just "shifted by one." So

$$L = \lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{a_n^2}{2} = \frac{\left(\lim_{n\to+\infty} a_n\right)^2}{2} = \frac{L^2}{2}.$$

Thus $2L = L^2$ and $L = 2$.

Alternatively we can notice that $a_n = 2^{1-\frac{1}{2^n}}$. Then

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} 2^{1-\frac{1}{2^n}} = 2^{\left(\lim_{n\to+\infty} 1-\frac{1}{2^n}\right)} = 2^{1-0} = 2.$$

4.2 Series

In this section we will discuss a particular type of sequence called a series. Series are powerful and flexible tools that show up in many places in mathematics; they are used to compute approximations, they underlie integrals, and they are often used to solve differential equations.

But at base, we can think of a series as a sort of a discrete version of the integral. The integral is "continuous", which means it adds up values from every point in the domain; a series will add up the values at only distinct, separated points in the domain.

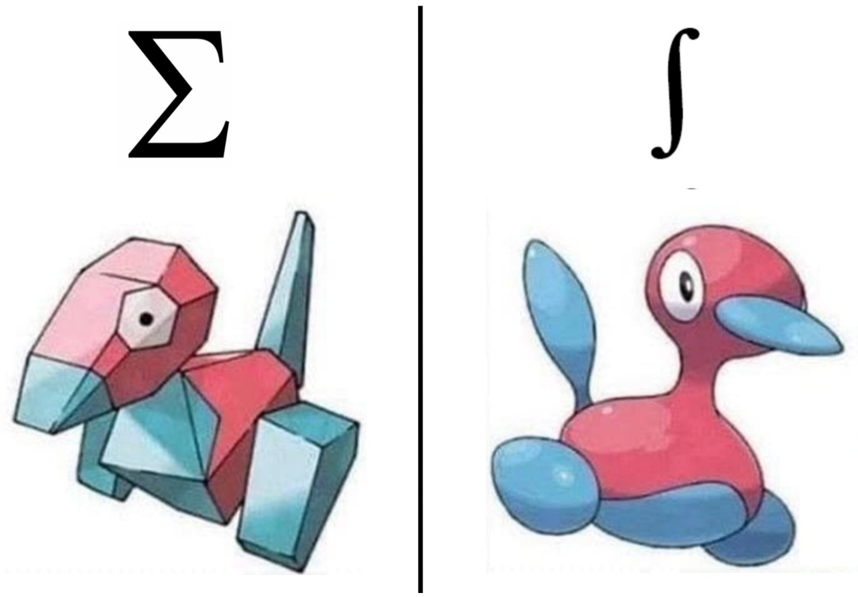


Figure 4.3: Meme courtesy of `@howie_hua` on Twitter

Definition 4.24. A *series* is a "sequence of partial sums." That is, a series is a sequence $(s_n)_{n=1}^{+\infty}$ where for some other real sequence (a_n) we have

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

If the sequence (s_n) is convergent and $\lim_{n\to+\infty} s_n = s$, then we say the series $\sum a_n$ converges to s, which is the *sum* of the series. We write

$$\sum_{n=1}^{\infty} a_n = s \qquad \text{or} \quad a_1 + a_2 + \cdots + a_n + \cdots = s.$$

If (s_n) is divergent, then the series is also divergent.

Example 4.25. A couple of the sequences we saw in the last section are "really" series.

- $1, 2, 3, \dots$ can be viewed as $\sum_{i=1}^{\infty} 1$.
- Any infinite decimal representation is really a series: we have

$$\pi = 3 + 1 \cdot 10^{-1} + 4 \cdot 10^{-2} + 1 \cdot 10^{-3} + \dots$$

Example 4.26. $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots$ is the series $\sum_{i=1}^{\infty} \frac{1}{2^i}$. We see that the partial sum $s_n = \sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$, and thus $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} 1 - \frac{1}{2^n} = 1 - 0 = 1$.

Remark 4.27. Notice that if the terms of a series are non-negative, then the sequence of partial sums is monotone increasing. Thus a series of positive terms either converges, or goes to infinity.

Example 4.28. The series $\sum_{n=1}^{\infty}(-1)^n$ has a sequence of partial sums $(-1, 0, -1, 0, \dots)$ and thus neither converges nor goes to infinity. But the terms are not all non-negative.

4.2.1 Telescoping Series and the Fundamental Theorem of Calculus

Series are the discrete version of integrals, but in general they're much harder to exactly compute. This is because we don't really have the Fundamental Theorem of Calculus—or at least, not in a useful way.

It's maybe worth thinking for a minute about what a discrete *derivative* would look like. In the continuous case, we say that the derivative approximates $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$. Even more informally, we say that $f'(x)$ is roughly the amount f increases if you increase x by one. But that's not quite right, because we're actually taking a limit as Δx gets very small, and so Δx can be much smaller than 1.

But in our discrete case, you can't have steps smaller than one. So the equivalent of the derivative would be $\dfrac{a_{n+1} - a_n}{(n+1) - (n)} = a_{n+1} - a_n$. This "difference quotient" is a perfectly useful calculation that shows up in a lot of contexts, but we won't talk about it much more in this course.

If we want to use the Fundamental Theorem of Calculus, we'd need to find a way to write the term inside our sum as a difference of two consecutive terms of a series. This is always technically possible, since your series itself is a sequence with the right differences of terms. But it's only rarely possible to view your terms as the difference quotients of a useful series.

Example 4.29. What is $\sum_{n=2}^{\infty} \frac{1}{n^2 - n}$?

Our sequence looks like

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots$$

which looks like it converges. By doing a partial fraction decomposition, we can write $\frac{1}{n^2-n} = \frac{1}{n-1} - \frac{1}{n}$. Then our partial sums are

$$\begin{aligned} s_n &= \sum_{i=2}^{n} \frac{1}{i-1} - \frac{1}{i} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots \\ &= 1 - \frac{1}{n}. \end{aligned}$$

Thus $\sum_{i=2}^{n} \frac{1}{n^2-n} = \lim_{n\to+\infty} 1 - \frac{1}{n} = 1$.

A series that works like this is called a *telescoping series*.

Example 4.30. Consider the series $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$. We can look at this as

$$\sum_{n=1}^{\infty} \log(n+1) - \log(n).$$

Then we can observe

$$\begin{aligned} s_k &= \sum_{n=1}^{k} \log(n+1) - \log(n) \\ &= \big(\log(k+1) - \log(k)\big) + \big(\log(k) - \log(k-1)\big) + \dots + \big(\log(3) - \log(2)\big) + \big(\log(2) - \log(1)\big) \\ &= \log(k+1) - \log(1). \end{aligned}$$

$\lim_{k\to\infty} s_k = \lim_{k\to\infty} \log(k+1) = \infty.$

Thus this sum diverges.

4.2.2 Series Rules

Just like with integrals, we can add series easily, and we can do scalar multiplication to them.

Proposition 4.31. *If $\sum a_n$ and $\sum b_n$ are convergent series, then*

- $\sum ca_n = c \sum a_n$.
- $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$.

And just like with integrals, we technically can multiply series together, but it's complicated and hard to use:

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k}$$

This operation is sometimes referred to as *convolution*. It it is too complicated to be terribly useful to us right now, but it often comes up in signal processing and more sophisticated approaches to differential equations.

4.2.3 Geometric Series

There's one more type of series that we can actually compute, which winds up being really important. These series don't *actually* telescope, but we can easily turn them into something that does.

Definition 4.32. A *geometric series* is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \dots$$

for some real numbers a and r.

Some people prefer to think of a geometric series as $\sum_{n=1}^{\infty} ar^n$. I'm one of them, actually, but your textbook isn't. It doesn't really matter which convention you use as long as you're consistent.

Can we add these series up? Let's cheat: we'll assume it's possible, and figure out what the sum should be. So let's start out assuming that $\sum_{n=1}^{\infty} ar^{n-1}$ converges to some number L. Then we have

$$\begin{aligned}
rL &= \sum_{n=1}^{\infty} ar^n = ar + ar^2 + r^3 + \dots \\
&= (-a) + \left(a + ar + ar^2 + ar^3 + \dots\right) = -a + \sum_{n=1}^{\infty} ar^{n-1} \\
&= -a + L \\
(r-1)L &= -a \\
L &= \frac{a}{1-r}.
\end{aligned}$$

For some questions, this answer is fine. We already argued in example 4.26 that $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. This is a geometric series with $a = r = \frac{1}{2}$, and thus

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1.$$

However, if we take this argument literally and don't do any more work, it suggests that $\sum_{n=1}^{\infty} 2^{n-1} = \frac{1}{1-2} = -1$, which is clearly absurd. (Well, usually. There's a trick called "regularization" that physicists use this for). But the basic idea is sound. First, we've shown that *if* the sume converges, it has to converge to $\frac{a}{1-r}$. And second, we can make the same argument a bit more carefully, paying attention to the limit, and getting something that actually works.

Let $s_n = \sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1}$. Then

$$\begin{aligned} rs_n &= \sum_{i=1}^{n} ar^i = ar + ar^2 + \cdots + ar^n \\ &= s_n - a + ar^n \\ (r-1)s_n &= a(r^n - 1) \\ s_n &= a\frac{r^n - 1}{r - 1}. \end{aligned}$$

We can think of this as a sort of anti-difference quotient: we have a closed-form formula for the nth partial sum.

If we take the limit as n goes to infinity, this diverges if $|r| \geq 1$. If $|r| < 1$, it converges, and we get the formula $\lim_{n \to +\infty} s_n = \frac{a}{1-r}$. We summarize this result:

Proposition 4.33. *If $\sum_{n=1}^{\infty} ar^{n-1}$ is a geometric series and $|r| < 1$, then*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

If $|r| \geq 1$ then the series diverges.

Example 4.34. What is $\sum_{n=1}^{\infty} \frac{2}{3^n}$? This is a geometric series with $a = \frac{2}{3}$ and $r = \frac{1}{3}$. (Note that $a = 2/3$ because a is the first term of the series.) So

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{2/3}{1 - 1/3} = 1.$$

Example 4.35. What is $\sum_{n=1}^{\infty} \frac{2^n}{3}$? This is a geometric series, this time with $a = 2/3$ and $r = 2$. Since $|2| \geq 1$ the series diverges.

We can also use this technique to turn infinite repeating decimals into integer fractions.

Example 4.36. Can we write $4.\overline{13}$ as a ratio of integers?

We have

$$4.\overline{13} = 4 + \frac{13}{100} + \frac{13}{100^2} + \frac{13}{100^3} + \ldots$$

After the first term we have a geometric series with $a = \frac{13}{100}$ and $r = \frac{1}{100}$, so the sum is

$$\frac{a}{1-r} = \frac{13/100}{99/100} = \frac{13}{99}.$$

Thus

$$4.\overline{13} = 4 + \frac{13}{99} = \frac{409}{99}.$$

Example 4.37. Does $\sum_{n=1}^{\infty} 3^{2n}2^{2-3n}$ converge or diverge?

This series looks like $\frac{9}{2} + \frac{3^4}{2^4} + \frac{3^6}{2^7} + \dots$. This is a geometric series with $a = \frac{9}{2}$ and $r = \frac{9}{8}$. Thus $|r| > 1$ and so the series diverges.

4.2.4 The Harmonic Series

There's one more series we can look at before we start building a general theory. This may be the single most important specific example we have.

Example 4.38. One of the most important series is the *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$. (It underlies among other things the Riemann zeta function which controls the distribution of prime numbers). Does it converge or diverge?

There's no really generalizable argument that applies here. But if $s_n = \sum_{i=1}^{n} \frac{1}{n}$ is the sequence of partial sums, then

$$\begin{aligned}
s_1 &= 1 > \frac{1}{2} \\
s_2 &= 1 + \frac{1}{2} > 2 \cdot \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 3 \cdot \frac{1}{2} \\
s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 4 \cdot \frac{1}{2}.
\end{aligned}$$

In particular, we see that $s_{2^{n-1}} > \frac{n}{2}$, and thus the sequence of partial sums increases without bound, and diverges to $+\infty$.

Remark 4.39. We will see that in some sense, the harmonic series is as small as it can get and still diverge.

Example 4.40 (Bonus Example)**.** The Kempner Series is the harmonic series, except we leave out every term where a 9 appears in the denominator. We claim that this series converges. (Yes, seriously. See also `http://www.smbc-comics.com/index.php?id=3777`).

We divide the series up according to the number of digits in the denominator. Among denominators with k digits, there are at most $8 \cdot 9^{k-1}$ since there are eight possibilities for the first digit (which cannot be 0 or 9) and 9 possibilities for the other digits (which cannot be 9). And each number is at least 10^{k-1}, so each term with k digits in the denominator is at most 10^{1-k}.

Then if we sum up all the terms with k digits in the denominator, we have $8 \cdot 9^{k-1}$ terms each of which is at most 10^{1-k} and so our sum is at most $\frac{8 \cdot 9^{k-1}}{10^{k-1}}$.

Now if we sum up the whole series, that's the same as summing up each set of k-digit denominators, and then summing all those sums. So we have

$$K \leq \sum_{k=1}^{\infty} 8 \frac{9^{k-1}}{10^{k-1}} = \sum_{k=1}^{\infty} 8 \left(\frac{9}{10}\right)^{k-1}.$$

This right-hand sum should look familiar; it's a geometric series. We have $a = 8$ and $r = \frac{9}{10}$, so the sum is

$$K \leq \sum_{k=1}^{\infty} 8 \left(\frac{9}{10}\right)^{k-1} = \frac{8}{1 - 9/10} = 80.$$

(A.J. Kempner first studied this series in 1914, and came up with the above argument. In 1979 Robert Baille showed that $K \approx 22.9$.)

Remark 4.41. In fact, if you take the harmonic series pick *any* string of digits, and remove terms with that string in the denominator, you get a convergent series, for basically the same reason.

4.3 The Divergence and Integral Tests

Now we can start building some general theoretical tools for understanding whether series converge.

4.3.1 The Divergence test

In the last section, we showed the harmonic series diverged by showing it was bigger than an infinite sum of a constant. This makes sense, because if you add the same number to itself infinitely many times, you will never get a finite amount. In fact, series can only converge if the terms get increasingly small as you go further into the series.

Proposition 4.42. *If* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n \to +\infty} a_n = 0$. *Thus if* $\lim_{n \to +\infty} a_n \neq 0$, *or if the limit does not exist, then* $\sum_{n=1}^{\infty} a_n$ *does not converge.*

Remark 4.43. The converse is not true. The divergence test can be used to show a series diverges; it cannot show that a series converges.

The divergence test winds up being a sort of first-pass filter. It lets us check that a series diverges really quickly, but can never tell us that a series converges.

Example 4.44. Consider the series $\sum_{n=1}^{\infty} 1$. We can see that $\lim_{n\to+\infty} 1 = 1 \neq 0$, so this series diverges. (We can see that in other ways by seeing that it must go to ∞.)

Example 4.45. Consider the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$. We can see that $\lim_{n\to+\infty} \frac{n}{n+1} = 1 \neq 0$. Thus this series diverges.

Example 4.46. Consider the series $\sum_{n=1}^{\infty}(-1)^n$. We can compute $\lim_{n\to\infty}(-1)^n$, but this limit does not exist. Thus by the divergence test, the series diverges (as we saw in example 4.28).

Example 4.47. The divergence test tells us nothing about the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. $\lim_{n\to+\infty} \frac{1}{n} = 0$, so we have no information. But we know that the harmonic series diverges by the argument in example 4.38.

This is a good example of how the divergence test can't show us a series converges. The harmonic series "passes" the divergence test: the terms go to zero. But that doesn't mean the series converges, and in fact it does not.

4.3.2 The Integral Test

So how can we tell that a series *converges*? Remember that we started this section with two principles. First, series are the discrete equivalent of integrals. Second, whenever possible, we want to convert discrete problems into continuous problems.

Example 4.48. Let's look at the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. The limit of the terms is $\lim_{n\to\infty} \frac{1}{n^2} = 0$, so the divergence test doesn't tell us anything. We can use a computer to calculate some experimental values: we compute that $\sum_{i=1}^{10} \frac{1}{i^2} \approx 1.55$ and $\sum_{i=1}^{1000} \frac{1}{i^2} \approx 1.64$. This makes it look like the series is converging; but can we prove it?

Let's draw a picture (figure 4.4). Let $f(x) = \frac{1}{x^2}$, and then the values of the sequence we're adding up are the points $f(n)$. Treat each of these points as the right endpoint of a rectangle of width one; then we see the integral of f from 1 to k is definitely larger than $\sum_{n=2}^{k} \frac{1}{n^2}$. (We did leave out the first term of the series, but that doesn't matter; since it's finite, it can't affect whether our series converges.)

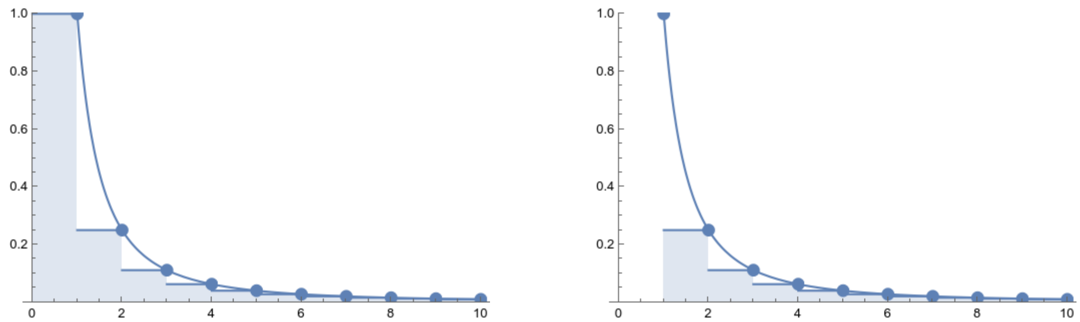


Figure 4.4: Removing the first rectangle changes the value of the integral, but can't affect whether it's finite or infinite.

Thus
$$\sum_{n=2}^{k} \frac{1}{n^2} \leq \int_1^k \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^k = 1 - \frac{1}{k}.$$
Taking the limit gives a right hand side of 1, and thus the sum $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is bounded and so must converge.

Remark 4.49. It turns out that the exact sum of this series is $\pi^2/6 \approx 1.64493$. This was first proven by Leonhard Euler in 1734, originally establishing his reputation. The proof is moderately complicated and requires a number of tools relating to power series, which we will discuss later in the course. (If you're interested, look up the "Basel Problem" on Wikipedia).

Example 4.50. Does the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converge?

We can use the same rough process and roughly the same picture we just did. By taking rectangles with *left* endpoints, we have
$$\sum_{n=1}^{k} \frac{1}{\sqrt{n}} \geq \int_1^k \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}|_1^k = \sqrt{k} - 1.$$
Taking the limit of both sides shows that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \geq \infty - 1$, and thus increases without bound.

We can build these types of argument into a general rule:

Proposition 4.51 (Integral Test)**.** *Suppose f is a continuous, positive, decreasing function on $[m, +\infty)$ for some m. Let $a_n = f(n)$. Then the series $\sum_{n=m}^{\infty} a_n$ converges if and only if $\int_m^{+\infty} f(x)\,dx$ converges. That is:*

- *If $\int_m^{\infty} f(x)\,dx$ converges then $\sum_{n=m}^{\infty} a_n$ converges.*

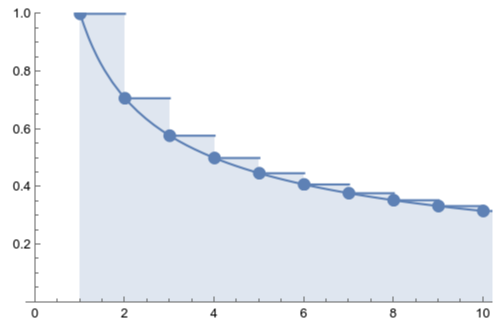


Figure 4.5: The series $\sum \frac{1}{\sqrt{n}}$ diverges, because $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges.

- *If $\int_m^\infty f(x)\,dx$ diverges then $\sum_{n=m}^\infty a_n$ diverges.*

Remark 4.52. Note that this doesn't tell us what the sum of the series is, just that it exists. In general, if we want to know the exact sum of a series we need a way to write a closed-form formula for the sequence of partial sums, which is hard. This is what I meant when I said that we don't have a useful equivalent to the fundamental theorem of calculus.

Most of the rest of the tools we'll develop in this class will only be used to establish that some series converges at all. This on its own can be useful, and we'll make it very useful in section 5 when we discuss Power Series and Taylor Series.

Example 4.53. Does $\sum_{n=1}^\infty \frac{2n}{n^2+1}$ converge?

Let $f(x) = \frac{2x}{x^2+1}$. Then f is clearly positive and continuous, and $f'(x) = \frac{2(x^2+1)-4x^2}{(x^2+1)^2}$ is negative so f is decreasing. So we can use the integral test.

$$\begin{aligned}\int_1^{+\infty} f(x)\,dx &= \lim_{t\to+\infty}\int_1^t \frac{2x}{x^2+1}\,dx \\ &= \lim_{t\to+\infty} \ln|x^2+1||_1^t = \lim_{t\to+\infty} \ln|t^2+1| - \ln|2| = +\infty.\end{aligned}$$

So $\int_1^{+\infty} f(x)\,dx$ diverges, and thus so does $\sum_{n=1}^\infty \frac{2n}{n^2+1}$.

Proposition 4.54. *The series $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.*

Proof. If $p = 1$ this is the harmonic series, and we know it diverges.

If $p \neq 1$ then $f(x) = \frac{1}{x^p}$ is a positive, decreasing, continuous function, so we can use the integral test. We have

$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \lim_{t\to+\infty}\int_1^t x^{-p}\,dx = \lim_{t\to+\infty} \frac{x^{1-p}}{1-p}|_1^t = \lim_{t\to+\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p}.$$

This converges precisely when $1 - p < 0$, precisely when $p > 1$. ☐

Example 4.55. Does $\sum_{n=1}^{\infty} \frac{n^2-n}{n^4+3n^3+n}$ converge?

We could technically use the integral test here. But that would, unfortunately, require us to integrate $\frac{x^2-x}{x^4+3x^3+x}$. This is definitely *possible* using a partial fractions argument, but it's not fun and it's not clean.

But we can try to argue something like this: We know that $n^2 - n < n^2$, and we know that $n^4 + 3n^3 + n > n^4$. This means that

$$\frac{n^2-n}{n^4+3n^3+n} < \frac{n^2}{n^4} = \frac{1}{n^2}$$
$$\sum_{n=1}^{\infty} \frac{n^2-n}{n^4+3n^3+n} < \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and that series converges by the p-series test. This implies that our original series also converges!

We did need the integral test to solve this last problem, because we used the integral test to prove the p-series test and used the p-series test there. But this argument allowed us to avoid having to integrate a difficult function.

4.4 The Comparison Tests

The integral test is powerful, and you can in theory answer nearly any question about positive series with the divergence test and the integral test combined. But in practice, the integral test can be really annoying to use, since we have to actually compute integrals. We want to use the work we've already done to avoid having to do more work.

We can do that by comparing new series to old series we've already worked out, systematizing the argument we made in example 4.55.

Proposition 4.56 (Comparison Test)**.** *Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. Then:*

- *If $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq b_n$ for all (sufficiently large) n, then $\sum_{n=1}^{\infty} b_n$ converges.*
- *If $\sum_{n=1}^{\infty} a_n$ diverges and $a_n \leq b_n$ for all (sufficiently large) n, then $\sum_{n=1}^{\infty} b_n$ diverges.*

(This Comparison Test is the discrete analogue of the Comparison Test for improper integrals we saw in section 3.1.3.)

Remark 4.57. Note that this only applies to series with positive terms. If a series has all positive terms, then either it goes to infinity or it converges (as a consequence of completeness, see section 4.1.1). Comparison rules out going to infinity, so the series has to converge.

But if we allow negative terms, there's a third option: oscillating between multiple values. For instance, $\sum \frac{1}{2^n}$ converges, and $-1 \leq \frac{1}{2^n}$ for all n, but $\sum_{n=1}^{\infty}(-1)$ does not converge.

We can rule out oscillation with something like the squeeze theorem, but that requires a lot more work. This comparison test isn't powerful enough to deal with non-positive series.

Using the comparison test requires us to have something to compare our series with. We usually use a power series $\sum n^p$ or a geometric series $\sum ar^{n-1}$.

Example 4.58. Does $\sum_{n=1}^{\infty} \frac{1}{n^3+n^2+n+1}$ converge?

We know that $n^3 \leq n^3 + n^2 + n + 1$, so $\frac{1}{n^3+n^2+n+1} \leq \frac{1}{n^3}$. Since $\sum \frac{1}{n^3}$ converges, we know that $\sum_{n=1}^{\infty} \frac{1}{n^3+n^3+n+1}$ converges by the Comparison Test.

Example 4.59. Does $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converge?

We could use the integral test, but we can also comment that $\ln n \geq 1$ for $n \geq 3$, so $\frac{\ln n}{n} \geq \frac{1}{n}$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we know that $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by the comparison test.

Example 4.60. Does $\sum_{n=1}^{\infty} \frac{1}{n!}$ converge or diverge?

The obvious comparison to make is to observe that $\frac{1}{n!} \leq \frac{1}{n}$. But this doesn't help us, because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity, and being less than something that goes to infinity doesn't tell us anything. But obviously $\frac{1}{n!}$ is *much* smaller than $\frac{1}{n}$, so we can probably come up with a better comparison.

For $n > 3$, we can work out that that $n! > n^2$: $n! = n(n-1)(n-2)\dots(3)(2)(1) \geq n(n-1)(n-2)$ and $(n-1)(n-2) > n$. Therefore $\frac{1}{n!} \leq \frac{1}{n^2}$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test. Thus the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges by the comparison test.

Alternatively: $n! > 2^{n-1}$, and thus $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$. But $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is a geometric series and converges since $r = 1/2 < 1$, so by the comparison test $\sum_{n=1}^{\infty} \frac{1}{n!}$ also converges.

Example 4.61. Does $\sum_{n=1}^{\infty} \frac{1}{n^3-n^2+1}$ converge?

This is a lot harder to work with. The obvious comparison is ti $\frac{1}{n^3}$, but it's not actually true that $\frac{1}{n^3-n^2+1} \leq \frac{1}{n^3}$. (In fact, $n^3 > n^3 - n^2 + 1$ for $n > 1$).

We can save it by fiddling with our comparison, and making the series we're comparing to bigger. Instead of $1/n^3$ we can try something like $2/n^3$. And it turns out that $n^3/2 < n^3 - n^2 + 1$ for $n > 1$, since $n^2 < n^3/2 + 1$. So $\frac{1}{n^3-n^2+1} \leq \frac{2}{n^3}$. This shows that $\sum_{n=1}^{\infty} \frac{1}{n^3-n^2+1}$ converges by the comparison test.

This argument worked, but it's fiddly and annoying and seems like it must be too complicated; we'd like to be able to say that $\frac{1}{n^3-n^2+1}$ looks "basically like" $\frac{1}{n^3}$ and so they behave the same. Fortunately there's a way to make that work out.

Proposition 4.62 (Limit Comparison Test)**.** *Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms, and $\lim_{n\to+\infty} \frac{a_n}{b_n}$ exists and is a finite, nonzero number. Then either both series converge, or both series diverge.*

Thus we have

$$\lim_{n\to+\infty} \frac{1/n^3}{1/(n^3-n^2+1)} = \lim_{n\to+\infty} \frac{n^3-n^2+1}{n^3} = 1,$$

and since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, so does $\sum_{n=1}^{\infty} \frac{1}{n^3-n^2+1}$.

Example 4.63. Does $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+n^2+1}$ converge?

We suspect we can compare this to $\frac{n^2}{n^3}$, or in fact to $\frac{1}{n}$, which has matching top degree. We check by calculating

$$\lim_{n\to+\infty} \frac{\frac{n^2+1}{n^3+n^2+1}}{1/n} = \lim_{n\to+\infty} \frac{n^3+n}{n^3+n^2+1} = \lim_{n\to+\infty} \frac{1+n^{-2}}{1+n^{-1}+n^{-3}} = 1.$$

This is a real number between 0 and $+\infty$. Thus, since $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge, by the limit comparison test $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+n^2+1}$ also diverges.

Example 4.64. Does $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt{n^5+n^3+n}}$ converge or diverge?

The numerator has the order of n and the denominator has the order of $n^{5/2}$, so we want to compare this to $\frac{n}{n^{5/2}} = \frac{1}{n^{3/2}}$. So we calculate

$$\begin{aligned}\lim_{n\to\infty} \frac{\frac{n+5}{\sqrt{n^5+n^3+n}}}{1/n^{3/2}} &= \lim_{n\to\infty} \frac{n^{5/2}+5n^{3/2}}{\sqrt{n^5+n^3+n}} \\ &= \lim_{n\to\infty} \frac{1+5/n}{\sqrt{1+1/n^2+1/n^4}} = 1.\end{aligned}$$

This is a real number in $(0,\infty)$, and thus the two series have the same convergence behavior.

Since $3/2 > 1$, by the p-series test we know that $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges. So by the limit comparison test, $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt{n^5+n^3+n}}$ converges.

Example 4.65. Does the series $\sum_{n=1}^{\infty} \frac{1}{3^n-2}$ converge or diverge?

We can't really use the regular comparison test here; the obvious point of comparison is $\sum \frac{1}{3^n}$, but $\frac{1}{3^n-2} > \frac{1}{3^n}$. But we can compute

$$\lim_{n\to\infty} \frac{1/(3^n-2)}{1/3^n} = \lim_{n\to\infty} \frac{3^n}{3^n-2} = \lim_{n\to\infty} \frac{1}{1-2/3^n} = 1.$$

Thus by the limit comparison test, $\sum_{n=1}^{\infty} \frac{1}{3^n-2}$ converges.

We *could* talk a lot more about limit comparison to a geometric series, but there'll be a better way to handle this in section 4.6 when we talk about the ratio test.

4.5 Non-Positive Series

So far we've only discussed series with all positive terms, and we have a pretty good handle on them: we use the integral test to work out some basic examples, and then solve others with the comparison tests.

Things get a little trickier when we want to talk about series that include negative terms. They can get very complicated, but we'll start off with an easy type of example.

4.5.1 Alternating Series

Definition 4.66. An *alternating series* is a series whose terms are alternately positive and negative: either all the odd terms are negative and the even terms are positive, or all the even terms are negative and all the odd terms are positive.

Example 4.67. Some alternating series are

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n+3} = -\frac{1}{4} + \frac{4}{5} - \frac{9}{6} + \frac{16}{7} - \dots$$

Every alternating series $\sum a_n$ looks like $\sum (-1)^n |a_n|$ or $\sum (-1)^{n-1}|a_n|$.

Alternating series are relatively easy to study, because they have such a regular pattern. Fundamentally, an alternating series will go up, and then down, and then up again, but not as high as at first. Each peak will be lower than the previous peak, and each low point will be higher than the previous low point, as wee see in figure 4.6, so the series much converge somewhere.

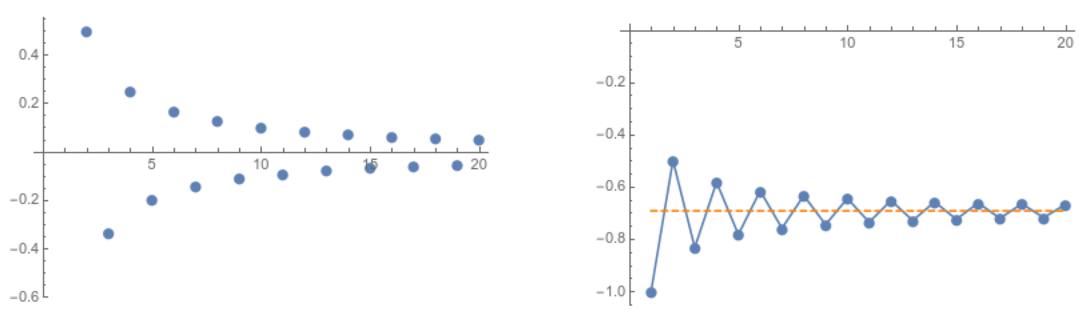


Figure 4.6: When we add up the terms of an alternating series, they oscillate up and down around the limit of the series

Proposition 4.68 (Alternating Series Test)**.** *If* $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ *is an alternating series such that* $b_{n+1} < b_n$ *for all (sufficiently large)* n*, and* $\lim_{n\to+\infty} b_n = 0$*, then the series is convergent.*

Sketch of Proof. The limit $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots \leq b_1$ is bounded above, and $(b_1 - b_2) + (b_3 - b_4) + \dots$ is increasing, so the sequence of even partial sums $(s_2, s_4, s_6, \dots)$ must converge to some limit. But s_{2n+1} has to be close to s_{2n}, so the entire sequence must converge. □

Example 4.69. The alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges by the alternating series test, since $\frac{1}{n+1} < \frac{1}{n}$ and $\lim_{n\to+\infty}\frac{1}{n} = 0$.

Example 4.70. The series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n+1}$ does not converge. The series is alternating, but the alternating series test does not apply because $\lim_{n\to+\infty}\frac{n}{n+1} = 1 \neq 0$. In fact, we see that $\lim_{n\to+\infty}(-1)^{n-1}\frac{n}{n+1}$ does not exist, so by the divergence test this series diverges.

Example 4.71. The series $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{n^4+2}$ converges. The sequence $\frac{n^3}{n^4+2}$ is decreasing, as we can see by taking the derivative of $f(x) = \frac{x^3}{x^4+2}$. Further, the limit is zero, so by the alternating series test the series converges.

The Alternating Series Test, combined with the Divergence Test means that we can test the convergence of (almost) any alternating series really easily. If the terms go to zero, it converges by the alternating series test; if the terms don't go to zero, it diverges by the divergence test.

Thus normally the divergence test is a necessary but not sufficient condition. For an alternating series specifically, it is both necessary and sufficient.

One other nice thing about alternating series is that we have a very good estimate of how close we are to the true sum. That means we can calulate estimates fairly easily, and know exactly how many terms we need to work out to be correct within our desired margin of error.

Proposition 4.72 (Alternating Series Estimation)**.** *If* $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ *is an alternating series that satisfies the hypotheses of the Alternating Series Test, then* $|s - s_n| \leq b_{n+1}$.

Sketch of proof. As we saw in figure 4.6, each term we add moves us *past* the limit. So our error at s_n has to be less than the size of the move we'll make by adding on the next term b_{n+1}. □

Example 4.73. Consider the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$. What is the error term in approximating the sum if we calculate the first ten terms?

The size of the error is smaller than the next term, which is the eleventh term, which is $\frac{1}{11}$. Thus $\sum_{n=1}^{10} \frac{(-1)^n}{n} \approx -.65$ is within $\frac{1}{11}$ of the infinite sum. In section 5.2.1 we will see that the exact sum of this series is $-\log 2 \approx -.69$.

Example 4.74. Consider the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$. What is the error term in approximating the sum if we calculate the first ten terms?

The size of the error is smaller than the next term, which is the eleventh term, which is $\frac{1}{121}$. Thus $\sum_{n=1}^{10} \frac{(-1)^n}{n^2} \approx -.818$ is within $\frac{1}{121}$ of the infinite sum, which turns out to be about $-.822\dots$.

Example 4.75. Consider the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+2n+1}$. How many terms do we have to calculate to get the answer to within $1/100$?

The ninth term has size $\frac{1}{9^2+18+1} = \frac{1}{100}$, so we need to compute the first eight terms. This gives approximately $-.1720$, while the true answer is approximately $-.1775$.

4.5.2 Absolute Convergence

The alternating series test allowed us to study one particular type of series with non-positive terms, but there are many non-positive series that aren't alternating. It's very difficult to study them in general, but the idea of absolute convergence allows us to mostly duck the question.

Definition 4.76. A series $\sum a_n$ is called *absolutely convergent* if $\sum |a_n|$ converges.

A series $\sum a_n$ is *conditionally convergent* if it is convergent but not absolutely convergent.

The series $\sum |a_n|$ is always non-negative, so we can use our tools from sections 4.3 and 4.4 to figure out whether this absolute-value series converges. But is that useful?

Theorem 4.77. *If $\sum a_n$ is absolutely convergent, then it converges.*

Proof. $0 \leq a_n + |a_n| \leq 2|a_n|$, and $a_n + |a_n| \geq 0$. We have $\sum_{n=1}^{\infty} 2|a_n|$ converges, so by comparision test $\sum_{n=1}^{\infty} a_n + |a_n|$ converges. But then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} a_n$$

is a difference of convergent series and so converges. □

Remark 4.78. The converse is not true! $\sum_{n=1}^{\infty} (-1)^n/n$ is convergent (by the alternating series test) but not absolutely convergent. This is why it's possible, and in fact relatively common, for a series to be conditionally convergent.

This theorem lets us study many sequences with positive and negative terms.

Example 4.79. The series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is absolutely convergent. We have $\left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so by the comparison test $\sum \left|\frac{\sin n}{n^2}\right|$ converges.

Example 4.80. The alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the alternating series test, but $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. So the series is conditionally convergent.

Example 4.81. The series $\sum_{n=1}^{\infty} \sin n$ diverges by the divergence test, since $\lim_{n\to+\infty} \sin n$ does not exist.

These are the three possible answers we can ever have: absolute convergence, conditional convergence, and divergence.

Example 4.82. We claim that $\sum_{n=1}^{\infty}(-1)^n/n^2$ converges absolutely. For $\sum_{n=1}^{\infty} |(-1)^n/n^2| = \sum_{n=1}^{\infty} n^{-2}$ which we know converges.

The main purpose of this is to take questions about series with some negative terms, and turn them into questions about series with positive real terms, so that our previous tests apply.

Example 4.83. Does the series $\sum_{n=1}^{\infty} \frac{\sin(n^2+e^n)}{n^2}$ converge?

We have that $\left|\frac{\sin(n^2+e^n)}{n^2-n}\right| = \leq \frac{1}{n^2}$, so by the comparison test this series converges absolutely. Thus it converges.

As one final note: absolutely convergent series are much nicer and easier to handle than series that are merely conditionally convergent.

Proposition 4.84. *If a series is absolutely convergent, then the sum doesn't depend on the order of the terms. (In particular, the sum of a series of positive numbers doesn't depend on the order of the terms).*

If a series is conditionally convergent but not absolutely convergent, then the sum does *depend on the order of the terms; and in fact by reordering the terms we can get essentially any sum we like.*

More precisely, the Riemann Series Theorem says that if $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent real series, then by reordering we can cause the sum to converge to any real number, or to diverge to $+\infty$ or $-\infty$.

Example 4.85. It's possible to compute that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$. But we also have

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots = \frac{\ln 2}{2}.$$

Proof. Suppose $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series of real numbers. Rewrite it as $\sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} c_n$ where the b_n are all the positive terms and the c_n are all the negative terms. If both of these sums converged, then the series would converge absolutely (since $\sum b_n + \sum c_n = \sum b_n + c_n = \sum |a_n|$); if one converged and the other diverged, then $\sum a_n$ would diverge. So $\sum b_n = \sum c_n = +\infty$.

Pick a target M. Arrange the sum as follows: include positive terms until the sum is above M. Then include negative terms until the sum is below M. Repeat, alternating, infinitely. The sum will oscillate around M and converge to M.

If we want the sum to approach $+\infty$, include positive terms until the sum is above 1, then a negative term, then positive terms until the sum is above 2, then a negative term, and so on. □

4.6 The Ratio and Root Tests

Once we know to look for absolute convergence, we can use the comparison test on any series, but we'd like to cut out some steps.

4.6.1 The Ratio Test

If we imagine comparing our series to a geometric series, we get the *ratio test*:

Proposition 4.86 (Ratio Test)**.** *If* $\sum_{n=1}^{\infty} a_n$ *is a series and* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$*, then:*

- *If* $L < 1$ *then the series* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*
- *If* $L > 1$ *then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Remark 4.87. If $\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ or does not exist, then the ratio test tells us nothing. We have to use some other test or technique.

This test tends to work well when our series looks "almost" geometric, meaning the terms have nth powers in them, or when the terms contain factorials. It works badly when the terms have additions and subtractions within them, or more generally when the terms look polynomial rather than exponential.

Example 4.88. Analyze the convergence of $\sum_{n=1}^{\infty} \frac{1}{n!}$.

Since it has a factorial, this is a natural place to apply the ratio test. We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1/(n+1)!}{1/n!}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1,$$

so by the ratio test this series converges absolutely.

Example 4.89. Analyze the convergence of $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

Again, there are factorials so we want to use the ratio test. We have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} \right| = \lim_{n\to\infty} \frac{(n+1)n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n.$$

It's maybe not immediately clear to us whether this converges, or to what. But we know that

$$\lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n = e$$

by our definition of e from section 1.2.1. Thus we have

$$\lim \left(\frac{n}{n+1} \right)^n = \frac{1}{\lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n} = \frac{1}{e} < 1.$$

So by the ratio test this series converges absolutely.

Example 4.90. What about $\sum_{n=1}^{\infty} \frac{r^n}{n!}$? For what r does it converge?

We still want to use the ratio test. We have

$$\lim_{n\to\infty} \left| \frac{r^{n+1}/(n+1)!}{r^n/n!} \right| = \lim_{n\to\infty} \frac{r}{n+1} = 0 < 1.$$

By the ratio test, this converges absolutely for any r.

Example 4.91. Now let $r > 0$ be a real number. Does $\sum_{n=1}^{\infty} \frac{n!}{r^n}$ converge or diverge?

This is similar but opposite to the previous problem. We have

$$\lim_{n\to\infty} \left| \frac{(n+1)!/r^{n+1}}{n!/r^n} \right| = \lim_{n\to\infty} \frac{n+1}{r} = +\infty > 1$$

so by the ratio test this diverges.

Example 4.92. Analyze the convergence of $\sum_{n=1}^{\infty} \frac{n^2+1}{2^n}$.

We compute

$$\lim_{n\to\infty} \left| \frac{((n+1)^2+1)/(2^{n+1})}{(n^2+1)(2^n)} \right| = \lim_{n\to\infty} \frac{n^2+2n+1}{(n^2+1)\cdot 2} = \frac{1}{2} < 1.$$

So by the ratio test this converges.

4.6.2 The Root Test

The Root Test is similar to the ratio test, but is sometimes slightly easier or harder to apply than the Ratio Test is.

Proposition 4.93 (Root Test). *If $\sum_{n=1}^{\infty} a_n$ is a series and $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, then:*

- *If $L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.*
- *If $L > 1$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.*

Remark 4.94. If $\lim_{n\to+\infty} \sqrt[n]{|a_n|} = 1$ or does not exist, then the root test tells us nothing. We have to use some other test or technique.

This is most useful when our series has an nth power of some polynomial involving n. The root test works well if each term is a perfect nth power, and poorly if we have something like $n2^n$ where some terms aren't covered by the exponent. The ratio test works well if our terms don't have any additions or subtractions in them, but do have exponents.

Example 4.95. Analyze $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5n+1}{2n+2}\right)^n$.

We have $a_n = \left(\frac{5n+1}{2n+2}\right)^n$ and thus $\sqrt[n]{|a_n|} = \frac{5n+2}{2n+2}$. So $\lim_{n\to+\infty} \sqrt[n]{|a_n|} = \frac{5}{2} > 1$ so the series converges absolutely.

Example 4.96. Analyze $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n^2+1}{3n^2+2n+1}\right)^n$

Our terms are perfect nth powers, so the root test seems natural. We compute

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{2n^2+1}{3n^2+2n+1}\right|^n} = \lim_{n\to\infty} \frac{2n^2+1}{3n^2+2n+1} = \frac{2}{3} < 1.$$

So by the Root Test this series converges absolutely.

Example 4.97. Analyze $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$.

Our terms are perfect nth powers, so we can try the root test. We compute

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n}{n+1}\right|^n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

so the root test doesn't tell us anything! We could try the ratio test, but it would be much harder to apply and would give the same answer—the root and ratio tests always give the same answer.

We could try either a comparison test or an integral test, but the integral seems nasty, and I'm not sure what to compare it to. And at this point we realize we forgot the first rule of series convergence: try the divergence test first! We have

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \neq 0.$$

So by the divergence test, this series diverges.

5 Power Series and Taylor Series

In this section we want to use what we've done with series in order to accomplish things. In particular, we can use series to define functions. And this allows us to work with a lot of functions that we've talked about in the past, but didn't have ways to compute. Functions like $\ln(x)$ and $\arctan(x)$ are easy to *say* but hard to compute, and the same applies for things like $\int e^{-x^2}\,dx$.

With series, we can write down formulas for these things that allow us to get good approximate answers to questions we care about. And as a bonus, we can replace a lot of the calculus we've done before with much simpler operations.

5.1 Power Series

We want to start by figuring out how to build a function out of a series. There are a few ways to do this, but the one we'll be studying is called a *power series*.

(Another important tool is the *Fourier series*, which is important to any sort of digital music or video. Unfortunately they're a little more complicated and we won't have time to talk about them. But if you're interested you can check out section 5.7, or I recommend this video by 3Blue1Brown.)

Definition 5.1. A *power series* is a series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

More generally, a *power series centered at a* is a series

$$\sum_{n=1}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

Because this series contains x, we can plug in a number for x and get a "regular" series, which either converges to a number or diverges. Note we adopt the convention $x^0 = 1$, and $(x-a)^0 = 1$, so that if we plug in $x = a$ our sum is just c_0.

We can think of a power series as an "infinitely long polynomial"; c_0 is the constant term, then c_1 is the coefficient of x^1, and in general c_n is the coefficient of x^n.

We've seen one important power series in disguise before.

Example 5.2. The series

$$1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

is a geometric series with $a = 1$ and $r = x$. We know from our study of geometric series that this converges if $|x| < 1$ and diverges if $|x| \geq 1$, and that when it converges, the sum is $\frac{1}{1-x}$.

A first important step in understanding a power series is figuring out when it converges. That is, if we plug in a value for x, does it converge to a real number, or does it diverge? This is equivalent to asking for the domain of the function, an idea we'll return to in section 5.2.

Example 5.3. For what x does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

Because of the nth powers, we use the ratio test. (Note that power series always have nth powers and so you basically always use the ratio test). Since $a_n = \frac{x^n}{n!}$, we have

$$L = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \lim \left|\frac{x^{n+1}n!}{(n+1)!x^n}\right| = \lim \left|\frac{x}{n}\right| = 0.$$

Since $L = 0 < 1$ for any value of x, this series converges absolutely for any real number x.

Example 5.4. For what x does $\sum_{n=0}^{\infty} n(x-2)^n$ converge?

Again, we use the ratio test, as we almost always do for power series. We have $a_n = n(x-2)^n$ and $a_{n+1} = (n+1)(x-2)^{n+1}$, so

$$L = \lim_{n\to+\infty} \left|\frac{(n+1)(x-2)^{n+1}}{n(x-2)^n}\right| = \lim_{n\to+\infty} \left|\frac{n+1}{n}(x-2)\right| = |x-2|.$$

The ratio test says a series converges if $L < 1$, so this power series converges when $|x-2| < 1$. For real numbers this is when $1 < x < 3$.

When $L > 1$ the series diverges, so if $x < 1$ or $x > 3$ this series diverges. But the ratio test doesn't tell us what happens when $L = 1$, so we have to look at those cases separately.

When $x = 3$ then our series is

$$\sum_{n=0}^{\infty} n \cdot 1^n = \sum_{n=0}^{\infty} n$$

which clearly diverges by the divergence test. Similarly, if $x = 1$ our series is

$$\sum_{n=0}^{\infty} n \cdot (-1)^n$$

which is an alternating series, but again diverges by the divergence test.

Some power series will converge for all x, like that last example. And some power series will diverge for any $x \neq 0$. But most of them look like this, and converge sometimes. But that sometimes will always follow the same specific pattern.

Theorem 5.5. *If $\sum_{n=0}^{\infty} c_n(x-a)^n$ is a power series, then exactly one of the following things occurs:*

- *The series converges only when $x = a$;*
- *The series converges for any real number x;*
- *There is a positive number R, called the* radius of convergence*, such that the power series converges for $|x-a| < R$ and diverges for $|x-a| > R$. Note this tells us nothing about what happens when $|x-a| = R$; we have to check those cases individually.*

Remark 5.6. This is another explanation for the language of "absolute" and "conditional" convergence. A power series will converge everywhere on the interior of its interval of convergence. It diverges everywhere outside the interval. On the boundary of the interval it may or may not converge, depending on the specific boundary point; thus, on the boundary it converges "conditionally."

(This can all generalize to complex numbers in a really important and interesting way, but we're not going to engage with that much in this course. But in the complex case, you can replace the word "interval" with "disk" in this remark.)

Definition 5.7. The *open interval of radius r centered at c*, is

$$(c-r, c+r) = \{x : |x-c| < r\}$$

the set of all points of distance *less than* r from the center c.

The *closed interval* of radius r centered at c is

$$[c-r, c+r] = \{x : |x-c| \leq r\}$$

the set of all points of distance at most r from the center c.

Note the closed interval contains its boundary points and the open interval does not. This is important!

Example 5.8. For what real x does $\sum_{n=0}^{\infty} \frac{(x-4)^n}{n}$ converge? What is the radius of convergence?

Guess what? We use the ratio test! We have

$$L = \lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{(x-4)^{n+1}/(n+1)}{(x-4)^n/n} \right| = \lim \left| \frac{(x-4)n}{n+1} \right| = \lim |x-4|$$

so the power series converges for $|x-4| < 1$, and thus the radius of convergence is 1. The power series converges absolutely on $(3, 5)$.

To find the real numbers where the series converges, we have to check 3 and 5 manually. For $x = 3$ we get the series $\sum \frac{(-1)^n}{n}$ which converges by the Alternating Series Test; and for $x = 5$ we get the series $\sum \frac{1}{n}$ which diverges. Thus the series converges on $[3, 5)$ in the real numbers.

Example 5.9 (recitation)**.** The Bessel function (of order 0) is critical to any physics done in cylindrical coordinates, and thus any physics that occurs on a cylinder. We saw it in section 3.3 as the solution to the differential equation $x^2 y'' + xy' + x^2 y = 0$, but it can also be given by the power series:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

What is the radius and interval of convergence?

We use the ratio test. We have $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, so

$$\begin{aligned}
\lim \left| \frac{a_{n+1}}{a_n} \right| &= \lim \left| \frac{x^{2n+2}/2^{2n+2}((n+1)!)^2}{x^{2n}/2^{2n}(n!)^2} \right| \\
&= \lim \left| \frac{x^{2n+2}}{x^{2n}} \frac{2^{2n}}{2^{2n+2}} \frac{(n!)^2}{((n+1)!)^2} \right| \\
&= \lim \frac{|x|^2}{4(n+1)^2} = 0.
\end{aligned}$$

Thus the Bessel function of order 0 converges absolutely for all real numbers x. We say the radius of convergence is ∞ and the interval is all reals, or $(-\infty, +\infty)$.

Example 5.10. What is the radius and interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt{n^2 + n}}?$$

Ratio test.

$$\lim \left| \frac{(-2)^{n+1} x^{n+1} / \sqrt{(n+1)^2 + n + 1}}{(-2)^n x^n / \sqrt{n^2 + n}} \right| = \lim 2|x| \frac{\sqrt{n^2 + 3n + 2}}{\sqrt{n^2 + n}} = 2|x|.$$

Thus by the ratio test the power series converges absolutely when $2|x| < 1$, or in other words when $|x| < 1/2$. The radius of convergence is $1/2$ and it converges on the open interval $(-{}^1\!/\!_2, {}^1\!/\!_2)$. Now we need to test endpoints.

When $x = 1/2$ then the series is

$$\sum \frac{(-1)^n}{\sqrt{n^2 + n}}.$$

This is an alternating series; the terms are decreasing and tend towards zero, so it converges.

When $x = -1/2$ then the series is $\sum \frac{1}{\sqrt{n^2+n}}$. We use the limit comparison test, and see that

$$\lim \frac{1/n}{1/\sqrt{n^2+n}} = \lim \sqrt{1+1/n} = 1$$

and thus $\sum \frac{1}{\sqrt{n^2+n}}$ has the same behavior as $\sum \frac{1}{n}$, and thus diverges.

So the real interval of convergence is $(-1/2, 1/2]$.

Example 5.11 (recitation)**.** What is the interval of convergence of

$$\sum_{n=0}^{\infty} \frac{n^2(x-1)^n}{7^{n+2}}?$$

Using the ratio test, we have

$$\lim \left| \frac{(n+1)^2(x-1)^{n+1}/7^{n+3}}{n^2(x-1)^n/7^{n+2}} \right| = \lim \frac{|x-1|}{7} \frac{(n+1)^2}{n^2} = \frac{|x-1|}{7}.$$

So the series converges absolutely when $|x-1| < 7$, and thus on the interval $(-6, 8)$. For the full interval we need to test the endpoints, at $x = -6$ and $x = 8$.

When $x = -6$ we have

$$\sum \frac{n^2(-7)^n}{7^{n+2}} = \sum (-1)^n \frac{n^2}{49}.$$

This is an alternating series, but the terms tend towards infinity and so by the divergence test it diverges.

Similarly, when $x = 8$ we have

$$\sum \frac{n^2 7^n}{7^{n+2}} = \sum \frac{n^2}{49}.$$

The terms tend towards infinity, so the series diverges by the divergence test.

Thus the real interval of convergence is $(-6, 8)$.

5.2 Power Series as Functions

Now that we understand how power series converge, we can see how to use them as as functions. In general, if we have a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, then we can define a function by $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$.

The domain of the function will be the interval of convergence of the power series. If the power series converges everywhere then the domain is all real numbers.

We already know how to express at least one function as a power series: by our geometric series argument, we know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$. We can use this fact to figure out how to express some other functions as power series as well.

Example 5.12. We can get new functions through composition, by plugging new formulas into the formula we already have. Thus

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty}(x^2)^n$$
$$\frac{1}{1+x^5} = \frac{1}{1-(-x^5)} = \sum_{n=0}^{\infty}(-x^5)^n = \sum(-1)^n x^{5n}.$$

Further, we can see that both of these must have the same interval of convergence as the original power series. The series for $\frac{1}{1-x}$ converges when $-1 < x < 1$, and thus the series for $\frac{1}{1-x^2}$ will converge when $-1 < x^2 < 1$, and this is precisely when $-1 < x < 1$. Similarly, $-x^5$ is in the interval $(-1,1)$ exactly when x is.

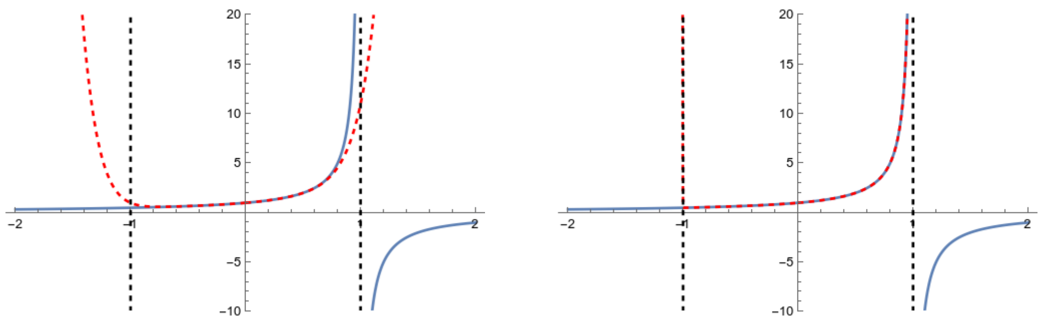


Figure 5.1: Left: the truncated power series $\sum_{n=0}^{10} x^n$. Right: The truncated power series $\sum_{n=0}^{1000} x^n$, which might as well be un-truncated

Remark 5.13. You may notice that $\frac{1}{1-x}$ has a domain bigger than $(-1,1)$. But the *power series* only converges on that smaller interval. There's something funky going on here because power series convergence has this symmetry requirement; if it converges at 2, it must also converge at -2. Since we *have* to have a divergence at $x=1$, we must also get divergence for $x>1$ and $x<-1$, as we see in figure 5.1.

Sometimes we have multiple options.

Example 5.14. How can we express $\frac{1}{x-3}$ as a power series? Since we want to write the denominator as $1-y$ for some expression y, we factor out a -3:

$$\frac{1}{x-3} = \frac{1}{-3}\cdot\frac{1}{1-x/3} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{-1}{3^{n+1}}x^n.$$

We know this will converge when $\left|\frac{x}{3}\right| < 1$, and thus when $|x| < 3$. So the interval of convergence is $(-3,3)$.

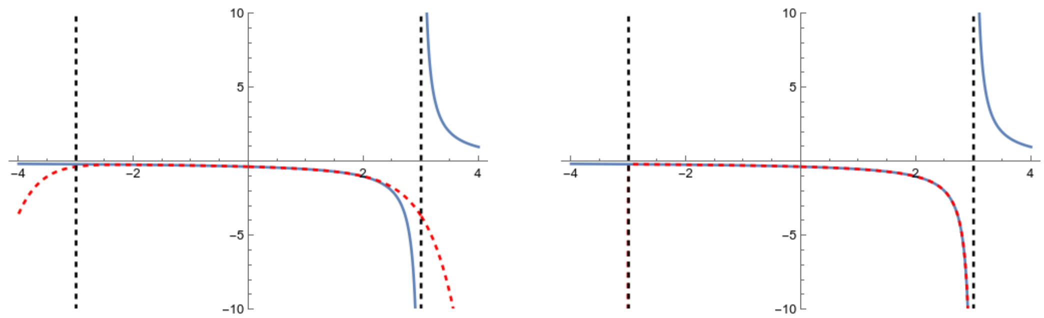


Figure 5.2: Power series for $\frac{1}{x-3}$ centered at 0.

Alternatively, we could write $\frac{1}{x-3} = \frac{1}{1-(4-x)}$. Then we have

$$\frac{1}{x-3} = \sum_{n=0}^{\infty}(4-x)^n = \sum_{n=0}^{\infty}(-1)^n(x-4)^n.$$

This is a power series with center 4, and it converges when $|4-x| < 1$ so it has radius of convergence 1.

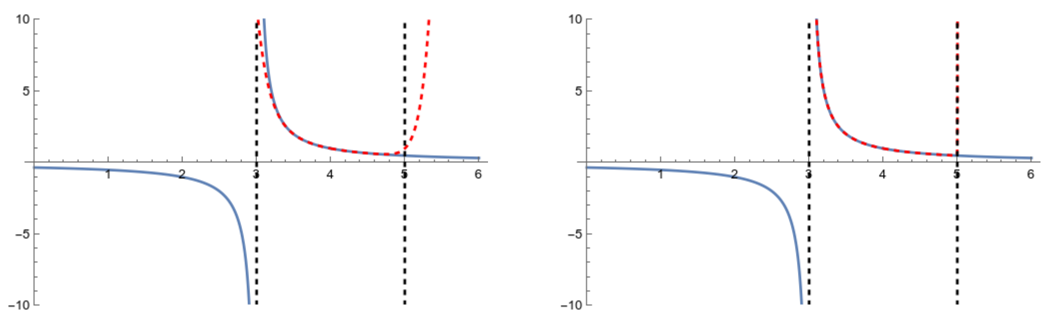


Figure 5.3: Power series for $\frac{1}{x-3}$ centered at 4. Notice how this power series has a totally distinct interval of convergence: it centers on on the other side of the asymptote, but still stops at that asymptote.

These are two different power series for the same function, but they're completely different. Each one has the largest radius of convergence it can without crossing the bad point at $x = 3$, but since they start in different places, on opposite sides of the asymptote, they have completely distinct intervals of convergence.

We can also do most basic algebra with power series.

Example 5.15. How can we express $\frac{x}{1-x}$ as a power series? This is just $x \cdot \frac{1}{1-x}$ and so

$$\frac{x}{1-x} = x\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}.$$

The interval of convergence is again $(-1, 1)$.

5.2.1 Calculus of power series

Recall that possibly the easiest functions for us to work with when we do calculus (or, indeed, almost anything else) are polynomials. It's easy to differentiate or integrate polynomials, and to calculate their outputs. The nice thing about power series is that they're basically fake polynomials, so they're almost as good.

Taking a derivative or integral of a single term of a power series is something we already know how to do, since each term is a polynomial (or, technically, a monomial). So for instance, we have

$$\frac{d}{dx} c_n(x-a)^n = c_n \cdot n(x-a)^{n-1}$$
$$\int c_n(x-a)^n \, dx = c_n \frac{(x-a)^{n+1}}{n+1} + C.$$

That means that we can do calculus on polynomials easily, just by working on each term. And it turns out the same thing works for power series.

Proposition 5.16. *If $\sum c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function defined by $f(x) = \sum c_n(x-a)^n$ is differentiable on $(a-R, a+R)$, and we have*

- $f'(x) = \sum_{n=0}^{\infty} c_n \frac{d}{dx}((x-a)^n) = \sum_{n=0} nc_n(x-a)^{n-1}$.
- $\int f(x)\,dx = \sum_{n=0}^{\infty} \left(\int c_n(x-a)^n\,dx\right) = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$.

Remark 5.17. This proposition tells us that after taking the derivative or integral, our power series still has the same radius of convergence. However, convergence at the *endpoints* may change.

Now that we have this extra tool we can find power series for more functions.

Example 5.18. Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we can differentiate both sides. The derivative of the left hand side is

$$\frac{d}{dx}(1-x)^{-1} = -1(1-x)^{-2} \cdot (-1) = \frac{1}{(1-x)^2}$$

and so

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}.$$

Note that we've dropped the $n = 0$ term because the derivative of x^0 is 0, and writing $0 \cdot x^{-1}$ would be silly.

Also note that we have

$$\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty}(n+1)x^n$$

so this is still a proper power series. For instance, we can use the ratio test to double-check that the radius of convergence is still $R = 1$.

Example 5.19. A subtler and more clever question: can we find a power series expression for $\ln(1+x)$?

We know that $\ln(1+x) = \int \frac{1}{1+x}\,dx$. We also know that $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n$. Integrating gives us that

$$\begin{aligned}
\ln(1+x) &= \sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1} \cdot (-1) + C \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + C.
\end{aligned}$$

To find the constant C we compute $\ln(1+0)$, since plugging 0 in on the right hand side will just yield C. (If this reminds you of what we did in section 3.3.5 to solve differential equations, good!) We know that $\ln(1) = 0$, so $C = 0$, and we have

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

The radius of convergence is still 1; we can see this using the ratio test or by recalling that our original power series has radius of convergence 1.

As with the geometric series, the radius of convergence can't possibly be larger than 1, since the function $\ln(1+x)$ has an asymptote at $x = -1$, as we see in figure 5.4.

In passing, this justifies my repeated claims that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln(2)$, by plugging $x = 1$ into this formula.

Example 5.20. Find a power series for $\arctan x$.

Again, we note that $\arctan x = \int \frac{1}{1+x^2}\,dx$, and we know that $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n$. Integrating gives

$$\begin{aligned}
\arctan x &= \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx + C \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C.
\end{aligned}$$

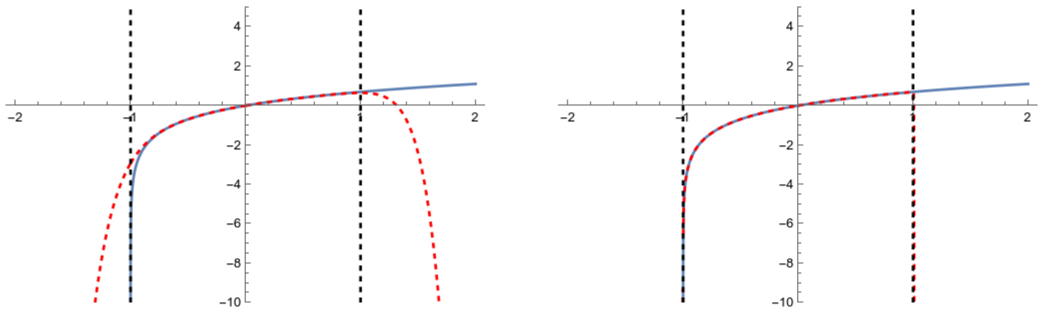


Figure 5.4: Left: the truncated power series $\sum_{n=1}^{10}(-1)^{n-1}\frac{x^n}{n}$. Right: The truncated power series $\sum_{n=1}^{1000}(-1)^{n-1}\frac{x^n}{n}$, which might as well be un-truncated

To find C we calculate $C = \arctan(0) = 0$, so we have

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}.$$

Again the radius of convergence is 1, as we see in figure 5.5. However, this time nothing about the function $\arctan(x)$ obviously forces this radius of convergence on us; it's purely an artifact of the way we set this up.

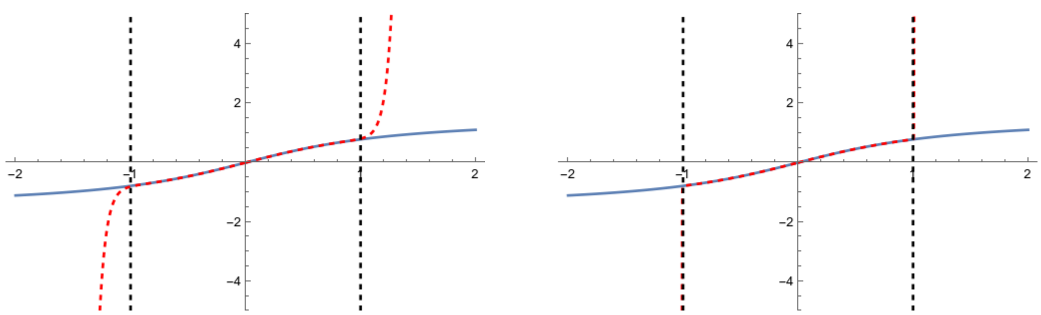


Figure 5.5: Left: the truncated power series $\sum_{n=0}^{10}(-1)^n\frac{x^{2n+1}}{2n+1}\sum_{n=1}^{10}(-1)^{n-1}\frac{x^n}{n}$. Right: The truncated power series $\sum_{n=0}^{500}(-1)^n\frac{x^{2n+1}}{2n+1}$, which might as well be un-truncated

Finally, these power series representations allow us to compute integrals that we either couldn't do or couldn't do easily before. We'll see more of this soon.

Example 5.21 (recitation)**.** What is $\int \frac{1}{1+x^6}$?

We could use a partial fractions decomposition, if we know that $1 + x^6 = (1 + x^2)(x^2 -$

$\sqrt{3}x+1)(x^2+\sqrt{3}x+1)$, but that's really unpleasant. Instead, we write

$$\frac{1}{1+x^6} = \sum_{n=0}^{\infty}(-x^6)^n = \sum_{n=0}^{\infty}(-1)^n x^{6n}$$
$$\int \frac{1}{1+x^6}\,dx = \sum_{n=0}^{\infty}\int (-1)^n x^{6n}\,dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+1}}{6n+1} + C.$$

This again converges for $|x| < 1$.

Example 5.22 (recitation)**.** What is $\int_3^4 \frac{1}{1-(x-4)^3}\,dx$?

Again we have

$$\frac{1}{1-(x-4)^3} = \sum_{n=0}^{\infty}(x-4)^{3n}$$
$$\int_3^4 \frac{1}{1-(x-4)^3}\,dx = \sum_{n=0}^{\infty}\int (x-4)^{2n}\,dx\bigg|_3^4$$
$$= \sum_{n=0}^{\infty}\frac{(x-4)^{3n+1}}{3n+1}\bigg|_3^4$$
$$= 0 - \sum_{n=0}^{\infty}\frac{(-1)^{3n+1}}{3n+1}$$

which converges for any n by the Alternating Series Test.

There are many other functions we wish we could integrate but can't; the most prominent example is e^{-x^2}, but there are others. Unfortunately, we don't have a power series for e^{-x^2}, and don't have an obvious way of obtaining power series for functions except by luck. But we can fix that.

5.3 Taylor Series

In the previous section we found power series for a number of familiar functions by starting with the power series for $\frac{1}{1-x}$, and then using clever algebraic or calculus manipulations to obtain forms for our new functions. But we'd like a more systematic way of approaching the problem, that doesn't rely on cleverness and luck.

Example 5.23. One particular function we'd like a power series representation for is e^x. Let's be optimistic and assume it has one, so we can write

$$e^x = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$$

for some collection of constants c_i.

As we saw in section 5.2, we can evaluate the power series at 0 (plug in $x = 0$) pretty easily. Plugging in $x = 0$ on the right hand side gives $c_0 + 0 + 0 + \cdots = c_0$, and so $c_0 = e^0 = 1$. But how can we determine the other constants?

Let's take the derivative of both sides. We get

$$e^x = \sum_{n=0}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \dots$$

and plugging in 0 for both sides gives $c_1 = e^0 = 1$. We can repeat the process; taking more derivatives gives

$$\begin{aligned} e^x &= 2c_2 + 6c_3 x + 12c_4 x^2 + \dots \\ e^x &= 6c_3 + 24c_4 x + 60c_5 x^2 + \dots \end{aligned}$$

and thus $2c_2 = 1, 6c_3 = 1$, and thus $c_2 = \frac{1}{2}$ and $c_3 = \frac{1}{6}$. Continuing this pattern, and more generally we have $c_n = \frac{1}{n!}$. Thus *if* we can represent e^x as a power series centered at 0, the power series must be

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We can generalize this to any function:

Theorem 5.24. *If f has a power series representation centered at a, that is, if*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for some sequence of constants c_n, then $c_n = \dfrac{f^{(n)}(x)}{n!}$ for each n, where $f^{(n)}(x)$ is the nth derivative of f at x.

Definition 5.25. We define the *Taylor series* of f centered at a to be

$$T_f(x,a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots$$

We sometimes say a Taylor series centered at 0 is a *Maclaurin series*, which we write

$$T_f(x,0) = T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \dots$$

Remark 5.26. It is *not true* that every function *can* be represented as a power series. If a function is not infinitely differentiable ("smooth") then it clearly doesn't have a power series, since all power series are smooth. Thus $|x|$ doesn't have a power series expansion that includes 0.

Not even all smooth functions have power series; we'll see an example in section 5.6.1. Functions that *can* be represented by power series are called "analytic."

But if a function can be represented by a power series, that power series is the Taylor series. Our next goal is to figure out when a function is equal to its Taylor series.

Definition 5.27. We call the truncated Taylor series the nth *Taylor polynomial* of f centered at a.

$$T_{F,k}(x,a) = T_k(x,a) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Remark 5.28. You might notice that $T_0(x,a) = f(a)$, and $T_1(x,a) = f(a) + f'(a)(x-a)$, which might look familiar from calculus 1 as the linear approximation to f near a. The Taylor polynomials in general are an expansion of this concept; T_1 is the best linear approximation we can make to f, and T_2 is the best quadratic appoximation we can make.

In computation and in modelling we often replace a function by its Taylor polynomial to make our lives easier. We'll use this for some applications later on in section 5.5.

The Taylor polynomials are not the same thing as our original function; they're approximations. so we can ask how much error this approximation has.

Definition 5.29. We define $R_k(x,a) = f(x) - T_k(x,a)$ to be the kth *remainder* of the Taylor series.

Now we can reframe our question—when is the Taylor series is equal to the original function?—by asking when the Taylor polynomials converge to the function. So $f = T_f(x,a)$ on some interval (b,c) if and only if $\lim_{k\to+\infty} R_k(x,a) = 0$ for any x in (b,c).

Fortunately there's a way to check this, related to the Mean Value Theorem. Recall from Calculus 1 that, if f is differentiable on an interval $[a,x]$, then there is a z in that interval such that

$$f'(z)(x-a) = f(x) - f(a) = R_0(x,a).$$

We can extend this result to include more derivatives, and get:

Proposition 5.30. *If f has enough derivatives on an interval I containing a, then for any x in I, there is a number z between x and a such that*

$$R_k(x,a) = \frac{f^{(k+1)}(z)}{(k+1)!}(x-a)^{k+1}.$$

Note that if we take $n = 0$ then we get the Mean Value Theorem.

Example 5.31. We'd like to show that the Taylor series for e^x we computed earlier actually gives us e^x. We have $f^{(k+1)}(z) = e^z$, so $R_k(x,0) = \frac{e^z}{(k+1)!}x^{k+1}$. Note that z depends on n, and z is between 0 and x, so assuming x is positive, we have

$$R_k(x,0) = \frac{e^z}{(k+1)!}x^{k+1} = \frac{x^{k+1}}{(k+1)!}e^z \leq \frac{x^{k+1}}{(k+1)!}e^x.$$

But as n goes to infinity, e^x doesn't change, and $\frac{x^{k+1}}{(k+1)!} \to 0$. So $\lim_{k\to+\infty} R_k(x,0) = 0$, and e^x is equal to its Taylor series for $x > 0$.

If x is negative, that argument doesn't quite work. But in that case we have $e^z < 1$ so we get

$$R_k(x,0) = \frac{e^z}{(k+1)!}x^{k+1} = \frac{x^{k+1}}{(k+1)!}e^z \leq \frac{x^{k+1}}{(k+1)!} \to 0.$$

Thus e^x is equal to its Taylor series for all x, and we can write

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and in particular

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots.$$

Example 5.32 (Recitation)**.** We can also ask for the Taylor series of e^x centered at another number, say $a = 1$. Each derivative is still e^x and thus e^3, and so we have

$$T(x,1) = \sum_{n=0}^{\infty} e^1 \frac{(x-1)^n}{n!}.$$

Is this actually equal to e^x? We compute

$$R_k(x,1) = \frac{e^z}{(k+1)!}(x-1)^{k+1}$$

which, for any fixed x and $|z| \leq x$ goes to 0 as k goes to infinity. So we have

$$e^x = \sum_{n=0}^{\infty} e^1 \frac{(x-1)^n}{n!}.$$

This is superficially different from the previous power series, but clearly the two series give the same function. The series centered at zero will be more efficient for computing with inputs near zero, and the series centered at 1 will be more efficient for inputs near 1.

This example also tells us another nice fact about e^x. Notice that if I plug 1 into this power series, I get e times what I would have gotten by plugging zero into the other power series. In general I can compute that

$$T_{x,a} = \sum_{n=0}^{\infty} e^a \frac{(x-a)^n}{n!} = e^a \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!}$$

which tells me that $e^x = e^a \cdot e^{x-a}$, which is the basic arithmetic rule for multiplying exponentials.

In a much crueler course, I could define e^x by its Taylor series, and then prove (or ask you to prove!) the rule that $e^{a+b} = e^a e^b$ by doing these sorts of algebraic manipulations.

Example 5.33. Can we find a power series for $f(x) = xe^x$ centered at 0? The obvious approach is to compute the Taylor series. We have

$$\begin{aligned}
f(x) &= xe^x & f(0) &= 0 \\
f'(x) &= e^x + xe^x & f'(0) &= 1 \\
f''(x) &= 2e^x + xe^x & f''(0) &= 2 \\
&\vdots & &\vdots \\
f^{(n)}(x) &= ne^x + xe^x & f^{(n)}(0) &= n.
\end{aligned}$$

Thus the Taylor series formula gives us

$$\sum_{n=0}^{\infty} \frac{n}{n!} x^n.$$

In order to remember that the constant term c_0 here is zero, I'll change the indexing to be from $n = 1$, and then I can see that $\frac{n}{n!} = \frac{1}{(n-1)!}$, and I get

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^n.$$

But there's another, easier way we could approach this. We already have a power series for e^x, so we can compute

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{1}{n!} x^n \\
xe^x &= x \sum_{n=0}^{\infty} \frac{1}{n!} x^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} x^{n+1}.
\end{aligned}$$

And these are in fact two ways of writing the same series, after we change the indexing around.

Example 5.34. Can we find a power series for $g(x) = e^{x^2}$?

Technically we can do this with the direct Taylor series method, but that's actually quite annoying. We can compute as before

$$\begin{aligned} g(x) &= e^{x^2} & g(0) &= 1 \\ g'(x) &= 2xe^{x^2} & g'(0) &= 0 \\ g''(x) &= 2e^{x^2} + 4x^2e^{x^2} & g''(0) &= 2 \\ g'''(x) &= 12xe^{x^2} + 8x^3e^{x^2} & g'''(0) &= 0 \\ g^{(4)}(x) &= 12e^{x^2} + 48x^2e^{x^2} + 16x^4e^{x^2} & g^{(4)}(0) &= 12 \end{aligned}$$

and we can kind of see a pattern here, but figuring out exactly what it is will be tricky and proving it works will be even trickier.

But there is, again, an easier way.

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{1}{n!} x^n \\ e^{x^2} &= \sum_{n=0}^{\infty} \frac{1}{n!} (x^2)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}. \end{aligned}$$

5.4 Computing Taylor Series

We want to compute some more Taylor series, both to get practice at doing the computation, and to have some more Taylor series to work with. We'll warm up with some simple examples.

Example 5.35. We can compute the Taylor series of a polynomial. In fact, if we take the series centered at 0, we get back exactly what we started with.

Let $f(x) = x^3 + 3x^2 + 1$. Then we have $f'(x) = 3x^2 + 6x$, $f''(x) = 6x + 6$, $f'''(x) = 6$, and $f^{(n)}(x) = 0$ for $n > 3$. Thus the Taylor series centered at 0 is

$$\begin{aligned} T_f(x, 0) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 \\ &= 1 + 0x + \frac{6}{2}x^2 + \frac{6}{6}x^3 = 1 + 3x^2 + x^3. \end{aligned}$$

Hopefully this is what you expected.

Probably more useful is the ability to write the Taylor series centered at a *different* point. If we take the Taylor series centered at 2, for instance, we have

$$\begin{aligned} T_f(x,2) &= f(2) + f'(2)x + \frac{f''(2)}{2}x^2 + \frac{f'''(2)}{6} \\ &= 21 + 24(x-2) + \frac{18}{2}(x-2)^2 + \frac{6}{6}(x-2)^3 \\ &= 21 + 24(x-2) + 9(x-2)^2 + (x-2)^3. \end{aligned}$$

If you multiply this out you will get your original polynomial back; but sometimes it is very useful to have a polynomial expressed in terms of $x-2$, say, instead of in terms of x.

Example 5.36. Let's consider the function $\ln x$. (We've computed a Taylor series for $\ln(1+x)$ but that's a bit awkward).

If $f(x) = \ln x$ then we have

$$\begin{aligned} f'(x) &= \frac{1}{x} & f''(x) &= \frac{-1}{x^2} \\ f'''(x) &= \frac{2}{x^3} & f^{(4)}(x) &= \frac{-6}{x^4} \\ &\dots \\ f^{(n)}(x) &= \frac{(-1)^{n-1}(n-1)!}{x^n} \end{aligned}$$

Thus if we wish to compute the Taylor series centered at 1, we have

$$\begin{aligned} \ln(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n \\ &= 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!1^n}(x-1)^n \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n \end{aligned}$$

This should look familiar (see figure 5.6); it's exactly the same thing as our previous series $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$, replacing x with $x-1$.

But wait, there's more! If we want to compute $\log(5)$, for instance, that power series doesn't work. But we can pick a new center for the power series and compute things there, and these power series will have different intervals of convergence, as we see in figure 5.7.

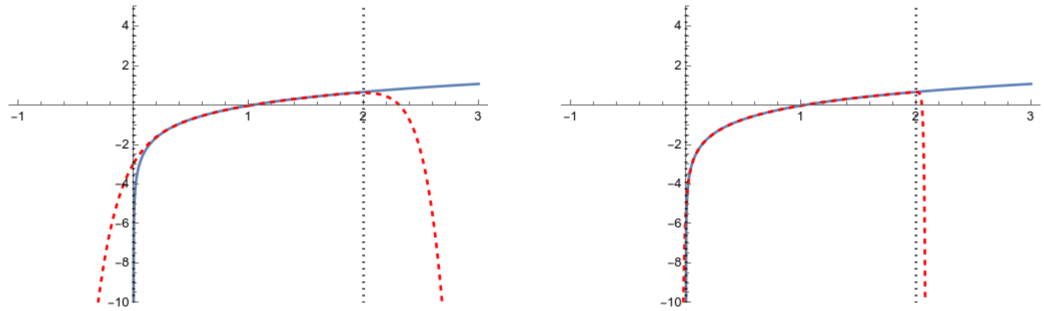


Figure 5.6: Graphs of the Taylor polynomials for $\ln(x)$ centered at 1, with 10 and 100 terms. Compare to the pictures in figure 5.4.

$$\begin{aligned}
\log(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(5)}{n!}(x-5)^n \\
&= \log(5) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!5^n}(x-5)^n \\
&= \log(5) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n}(x-5)^n.
\end{aligned}$$

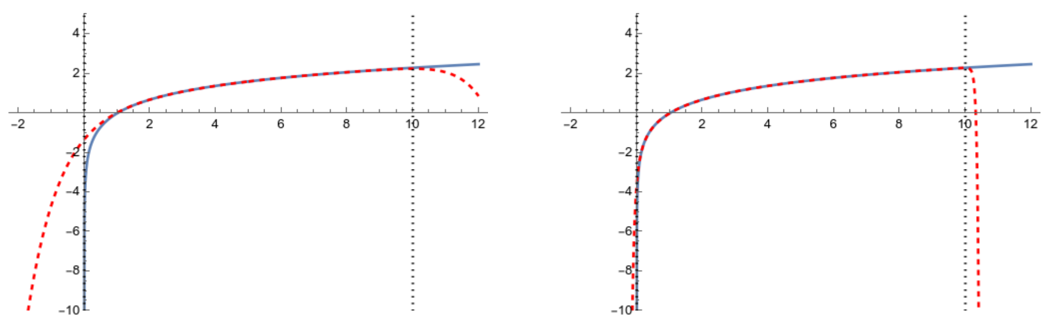


Figure 5.7: Graphs of the Taylor polynomials for $\ln(x)$ centered at 5, with 10 and 100 terms.

An application of the ratio test will show that this has radius of convergence 5, and we can see it converges conditionally on the boundary; in particular it will converge at 10 (by the alternating series test) and diverge at 0 (where it will just be the harmonic series). This behavior is in fact what we should expect: we know the series won't converge at 0 since $\log(0)$ is undefined, but we expect the series to converge everywhere it "can".

5.4.1 Trigonometry and Exponentials

While the most mysterious function we've been dealing with is e^x, we also would like to be able to compute $\sin x$ and $\cos x$.

First we'll compute the Maclaurin series for $\sin x$. Computing a few derivatives gives us

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= \cos(x) & \frac{d}{dx}\sin(x)\mid_0 &= 1\\
\frac{d^2}{dx^2}\sin(x) &= -\sin(x) & \frac{d^2}{dx^2}\sin(x)\mid_0 &= 0\\
\frac{d^3}{dx^3}\sin(x) &= -\cos(x) & \frac{d^3}{dx^3}\sin(x)\mid_0 &= -1\\
\frac{d^4}{dx^4}\sin(x) &= \sin(x) & \frac{d^4}{dx^4}\sin(x)\mid_0 &= 0
\end{aligned}$$

And since this pattern will repeate, we get that $\frac{d^{2n}}{dx^{2n}}\sin x = (-1)^n \sin x$, and $\frac{d^{2n+1}}{dx^{2n+1}}(\sin x) = (-1)^n \cos x$. Since $\sin(0) = 0$ and $\cos(0) = 1$, the Maclaurin series is

$$T(x,0) = 0 + x - 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} - 0 - \frac{x^7}{7!} + \dots \qquad = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Computing the remainder gives

$$\begin{aligned}
R_{2n}(x,0) &= \frac{(-1)^n \cos z}{(2n+1)!}x^{2n+1}\\
|R_{2n}(x,0)| &= \left|\frac{(-1)^n \cos z}{(2n+1)!}x^{2n+1}\right| \leq \frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

since $|\cos z| \leq 1$, and this tends to zero as n tends to infinity. So $\sin x$ is equal to its Maclaurin series, and we have

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

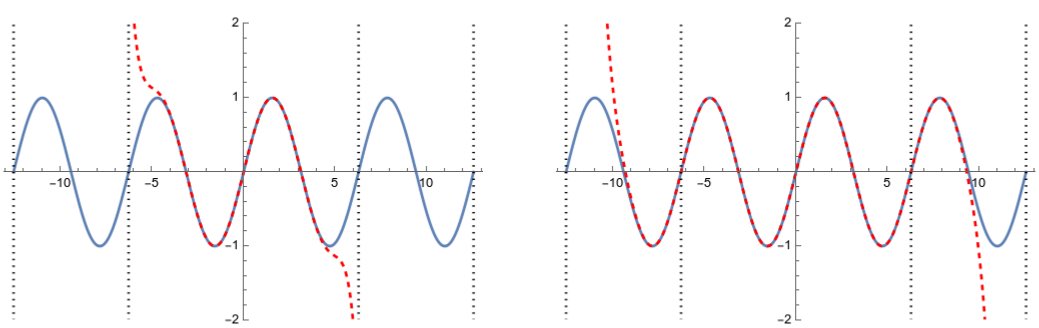


Figure 5.8: Graphs of the Taylor polynomials for $\sin(x)$ centered at 0, with 5 and 11 terms. Dashed asymptotes at multiples of 2π. Notice how quickly this converges near zero!

We also want a Maclaurin series for $\cos x$. We could compute it as we did before, but there's an easier way; $\cos x = (\sin x)'$, so

$$\begin{aligned}
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!} \\
&= 1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \dots
\end{aligned}$$

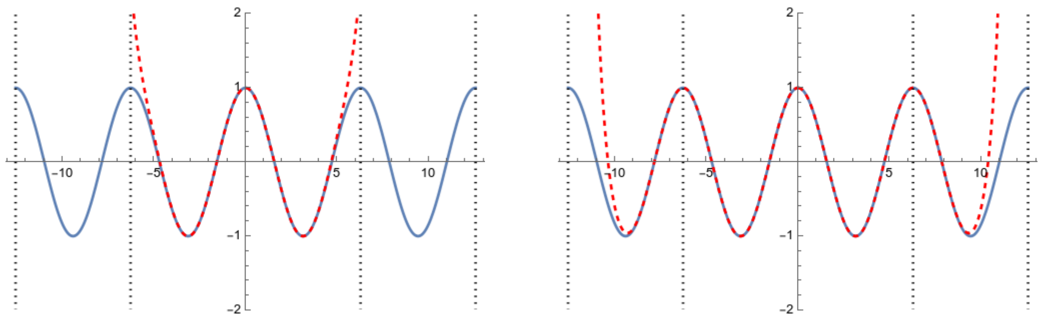


Figure 5.9: Graphs of the Taylor polynomials for $\sin(x)$ centered at 0, with 5 and 11 terms. Dashed asymptotes at multiples of 2π. Notice how quickly this converges near zero!

Though this is less important, sometimes we want to know things like the Taylor series for $x \sin x$. Again we'd rather not do this by computing derivatives, because that's hard. But we can find this Taylor series by doing power series algebra:

$$\begin{aligned}
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\
x \sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)!} = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \dots
\end{aligned}$$

5.4.2 The Binomial Series

Another important and widely applicable example is the *binomial series*, which is the Maclaurin series expansion for $f(x) = (1+x)^\alpha$. We can calculuate that

$$\begin{aligned}
f(x) &= (1+x)^\alpha & f(0) &= 1 \\
f'(x) &= \alpha(1+x)^{\alpha-1} & f'(0) &= \alpha \\
f'''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} & f''(0) &= \alpha(\alpha-1) \\
f'''(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} & f'''(0) &= \alpha(\alpha-1)(\alpha-2) \\
&\vdots & &\vdots \\
f^{(n)}(x) &= \alpha(\alpha-1)\dots(\alpha-n+1)(1+x)^{\alpha-n} & f^{(n)}(0) &= \alpha(\alpha-1)\dots(\alpha-n+1) \\
&= \frac{\alpha!}{(\alpha-n)!}(1+x)^{\alpha-n} & &= \frac{\alpha!}{(\alpha-n)!}
\end{aligned}$$

So we get the formula

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha!}{(\alpha-n)!n!} x^n.$$

We sometimes use the notation

$$\binom{\alpha}{n} = \frac{\alpha!}{(\alpha-n)!n!}$$

which we read "α choose n"; if α is a positive integer this represents the number of ways to choose n things out of α choices. Then we can write

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

By the ratio test, this power series converges when $|x| < 1$.

This series is called the binomial series, and is used very, very often to do numerical approximations, and especially in physics applications.

Notice that if α is a positive integer this is just the usual polynomial expansion. If α is an integer then $\binom{\alpha}{\alpha+1} = 0$, and so we get formulas like

$$(1+x)^5 = \sum_{n=0}^{5} \binom{5}{n} x^n = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5.$$

Example 5.37. What is $(3+x)^3$? We need to get a formula that looks like $(1+y)^3$, and factoring out a 3^3 gives us

$$(3+x)^3 = 3^3(1+x/3)^3.$$

Then the binomial series gives us

$$\begin{aligned}(3+x)^3 &= 27 \cdot (1+x/3)^3 \\ &= 27 \sum_{n=0}^{\infty} \binom{3}{n} \left(\frac{x}{3}\right)^n \\ &= 27 \left(1 + 3 \cdot \frac{x}{3} + 3 \cdot \frac{x^2}{9} + \frac{x^3}{27}\right) \\ &= 27 + 27x + 9x^2 + x^3.\end{aligned}$$

Example 5.38. What is $\sqrt[3]{1+x^2}$? This is the binomial series with $\alpha = 1/3$. So the Binomial Series tells us:

$$\begin{aligned}\sqrt[3]{1+x^2} &= \sum_{n=0}^{\infty} \binom{1/3}{n} x^{2n} \\ &= 1 + \frac{1}{3}x^2 + \frac{(1/3)(-2/3)}{2!}x^4 + \frac{(1/3)(-2/3)(-5/3)}{3!}x^6 + \dots \\ &= 1 + \frac{x^2}{3} - \frac{x^4}{9} + \frac{5x^6}{81} - \dots\end{aligned}$$

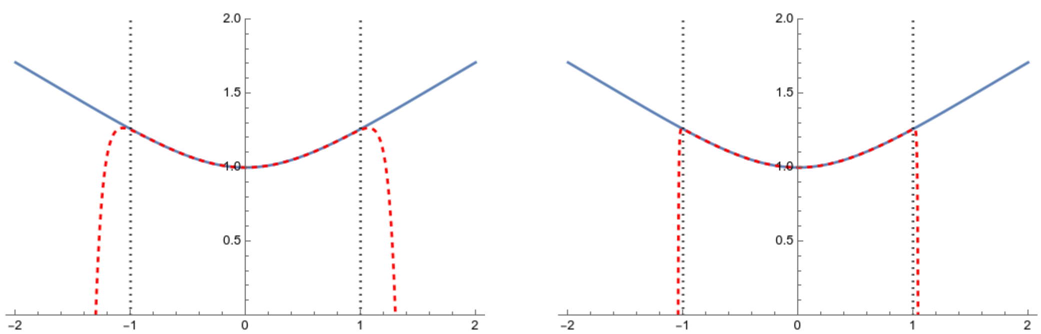


Figure 5.10: Graphs of the Taylor polynomials for $\sqrt[3]{1+x^2}$ centered at 0, with 20 and 200 terms. Again, notice how quickly this converges.

Thus we can estimate, for small x, that $\sqrt[3]{1+x^2} \approx 1 + \frac{x^2}{3}$. We would approximate that $\sqrt[3]{1.3} \approx 1 + \frac{.09}{3} = 1.03$. In fact Mathematica tells us that $\sqrt[3]{1.3} \approx 1.09139$, so that's pretty good.

5.5 Applications of Taylor Series

Now that we have power series representations of a bunch of functions, we can use them to calculate limits and integrals and other messy calculus things, and in general we can use them to do lots of cool things.

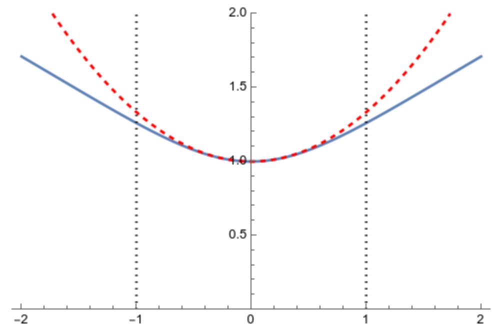


Figure 5.11: Graphs of the quadratic Taylor polynomial for $\sqrt[3]{1+x^2}$ centered at 0, which is just $1 + x^2/3$. That's already enough to get us a really good approximation.

5.5.1 Calculating constants

This is quick, but important. We all know that $\pi \approx 3.14$ and $e \approx 2.7$, but where do these numbers come from?

I've mentioned before that

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Summing the first five terms, though $n = 4$, gives

$$e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = \frac{65}{24} \approx 2.708$$

summing the first nine terms gives $\frac{109601}{40320} \approx 2.71828$.

Slightly trickier is finding π. The simplest way we have to do this is observing that $\arctan(1) = \pi/4$, and then computing

$$\pi = 4\arctan(1) = 4\sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}.$$

This series converges *very slowly*. Summing through $n = 10$ gives $\pi \approx 3.23232$; summing through $n = 100$ gives $\pi \approx 3.15149$. After a few hundred terms we see 3.14 show up, and at a thousand we get $\pi \approx 3.14259$.

There are much better series for calculating numerical approximations of π. But this one was good enough for a particularly stubborn gentleman named Abraham Sharp to compute π to 71 digits in 1699. By hand.

5.5.2 Approximating functions

The primary use of Taylor series is to conduct approximate calculations of functions we can't or don't want to calculate exactly. This means that when we're doing something like trying to understand a physical or economic model or solve a differential equation, we can pretend our functions are all polynomials, which are a lot easier to analyze.

Example 5.39. What is $\sqrt[n]{1+x}$ when x is small?

When we look at this we should immediately think of the binomial series (with $\alpha = 1/n$). Thus

$$\begin{aligned}\sqrt[n]{1+x} &= \sum_{k=0}^{\infty} \binom{1/n}{k} x^k = 1 + \frac{1}{n}x + \frac{(1/n)((1-n)/n)}{2}x^2 + \dots \\ &\approx 1 + \frac{x}{n}.\end{aligned}$$

Note that this approximation works better when x is small. But we can, as a rule of thumb, approximate $\sqrt[n]{2} \approx \frac{n+1}{n}$.

Example 5.40. Approximate $\sqrt[5]{x}$ near 32 to degree two.

We have two options. The first is to use the binomial series approximation. It looks like we want to approximate $(32+x)^{1/5}$, which isn't quite the binomial series which comes from $(1+x)^{\alpha}$; but we can factor out a two and get somtehing that works quite well:

$$\begin{aligned}\sqrt[5]{32+x} = 2(1+\tfrac{x}{32})^{1/5} &= 2\sum_{n=0}^{\infty}\binom{1/5}{n}x^n \\ &= 2 + \frac{2}{5}(x/32) - \frac{4}{25}(x/32)^2 + \dots\end{aligned}$$

(Note this converges when $|x/32| < 1$ and thus when $-32 < x < 32$). So we can approximate

$$\sqrt[5]{32+x} \approx 2 + \frac{x}{80} - \frac{x^2}{6400}.$$

This first approach is very common, and the reason we don't much mind that the binomial series approximation is specifically centered at 1. If we want to approximate $(r+x)^{\alpha}$ we can view this as $r^{\alpha}(1+x/r)^{\alpha}$ and then use the binomial series approximation.

Alternately, we could compute the Taylor polynomial anew, centered at 32:

$$\begin{aligned} f(x) &= \sqrt[5]{x} & f(32) &= 2 \\ f'(x) &= \frac{1}{5}x^{-4/5} & f'(32) &= \frac{1}{80} \\ f''(x) &= \frac{-4}{25}x^{-9/5} & f''(32) &= \frac{-1}{3200}\end{aligned}$$

and thus

$$\begin{aligned}\sqrt{x} &= \sum_{n=0}^{\infty}\frac{f^{(n)}(32)}{n!}x^n \\ &\approx 2 + \frac{x}{80} - \frac{x^2}{3200 \cdot 2}.\end{aligned}$$

Either way, we can estimate $\sqrt[5]{36} \approx 2 + \frac{4}{80} - \frac{16}{6400} = 2 + \frac{1}{16} - \frac{1}{800} \approx 2.06$.

5.5.3 Limits

Taylor series can make computing limits very easy. Heuristically when we calculate a limit we tend to ask "how many times" the top and bottom go to zero; we can see L'Hospital's rule as a way of calculating this. But working with Taylor series makes this idea precise.

Example 5.41. What is $\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{x^3}$?

We could use L'Hospital's rule here three times, and we did that in section 1.6. But we can also approach this with Taylor series. We know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$, so this is

$$\lim_{x\to 0} \frac{\frac{x^3}{3!} + \frac{x^4}{4!} + \dots}{x^3} = \lim_{x\to 0} \frac{1}{3!} + \frac{x}{4!} + \frac{x^2}{5!} + \dots = \frac{1}{6}.$$

Example 5.42. What is $\lim_{x\to 0} \frac{\sin x}{x}$?

With the same trick, we have

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\ \frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots \\ \lim_{x\to 0} \frac{\sin x}{x} &= \lim_{x\to 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \dots = 1. \end{aligned}$$

Similarly, we have

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) - 1}{x^2} = \lim_{x\to 0} \frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \dots = \frac{1}{2}.$$

Remark 5.43. In some sense, this what L'Hôpital's Rule is "really" doing. When the top and bottom are both zero, that means the constant terms of each power series are zero. We take the derivative to shift both power series over one place and then try comparing the linear terms. Then the quadratic, etc.

Notice that this is very like how we handle all limits as $x \to \infty$. In that case, we only have to care about the highest-degree term, and we can ignore all the others. Here, if $x \to a$, we only have to care about the *lowest*-degree term of the Taylor expansion around a.

5.5.4 Integration with Taylor Series

We can also use Taylor series to make difficult integrals easy.

Example 5.44. What is the integral of $x^6 \cos x$?

We can do this with integration by parts, but it's tedious. Instead, we calculate:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}$$
$$x^6 \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+6}}{2n!}$$
$$\int x^6 \cos x \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+7}}{(2n+7)(2n!)} + C.$$

Example 5.45 (Recitation)**.** There are some integrals that simply cannot be computed by normal means. I'v ementioned a few times that we can't represent $\int e^{-x^2} \, dx$ with "elementary" functions. But the integral is very important; any time you're dealing with, for instance, a normal distribution, the integral of e^{-x^2} is lurking in the background.

With our new techniques this is easy to handle:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$
$$\int e^{-x^2} \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(n!)} + C.$$

Thus we can compute, for instance, that

$$\int_0^1 e^{-x^2} \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(n!)} \Big|_0^1$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(n!)} \approx .75.$$

There are still interesting questions in actually computing things with this; our answers are written in terms of *infinite series* and we still need ways to approximately sum those series. But this gives us a way to answer the questions at all.

Remark 5.46. A technique of complex analysis called "contour integration" tells us that $\int_{-\infty}^{+\infty} e^{-x^2} \, dx = \sqrt{\pi}$. (I told you π shows up everywhere for no reason). From this fact it's not too hard to show that $\int_{-\infty}^{+\infty} e^{-x^2/2} \, dx = \sqrt{2\pi}$, and thus that $\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = 1$.

This is why the standard bell curve (with mean zero and standard deviation 1) is given by the probability density function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$; the total probability of *something* happening has to be exactly one. We can generalize this to a normal distribution with mean μ and standard deviation σ, which has probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

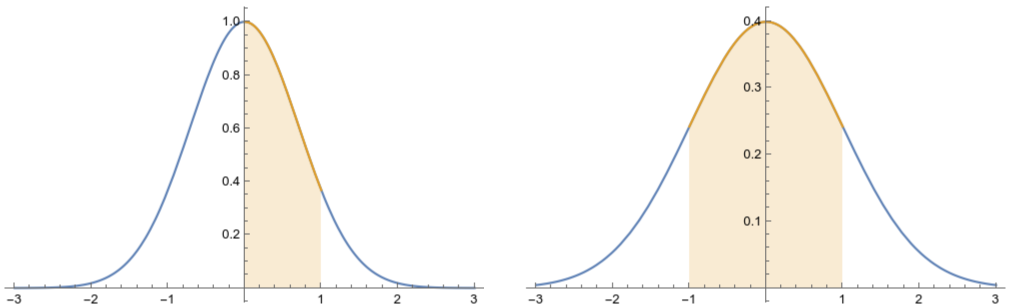


Figure 5.12: Left: $\int_0^1 e^{-x^2}\,dx$. Right: The probability of landing within one standard deviation of the mean.

This still has total integral one. But there's still no elementary antiderivative to this function, so if we want to compute, say, the probability of an output between -1 and 1 as in figure 5.12 we still need to use some sort of Taylor series argument to approximate the probability.

5.5.5 Maxima and Minima

Taylor series give us another way to think about what we're doing when we look for maxima and minima of functions. We can approximate any (reasonable) function with Taylor polynomials. So first think about what happens when we approximate with a linear function. If the linear function is sloping upwards or downwards, then our function doesn't have a maximum or a minimum. And that's basically what we said in calculus 1. At a maximum, T_1 needs to be constant.

But now think about the second-order Taylor approximation, which will be a parabola. Since the first-order Taylor approximation was constant, we know we'll be at the vertex of a parabola. But this parabola will either open up or down. If our function is approximately an upwards parabola, we have a minimum; if our function is approximately a downwards parabola, we have a maximum. And what determines if the parabola opens up or down? If the leading quadratic term has a positive coefficient, it will open up, and if the leading quadratic term has a negative coefficient, then it will open down. And that's exactly the second derivative test from calculus 1.

But now we can do more! If the second derivative is zero, then the second derivative test doesn't give us any information. And that's because the second-order Taylor polynomial T_2 is just a horizontal line: it doesn't tell us enough. But we can compute the Taylor series further out.

Example 5.47. Suppose we want to look at $f(x) = \cos(x^2)$. We see that $f'(x) = -2x\sin(x^2)$

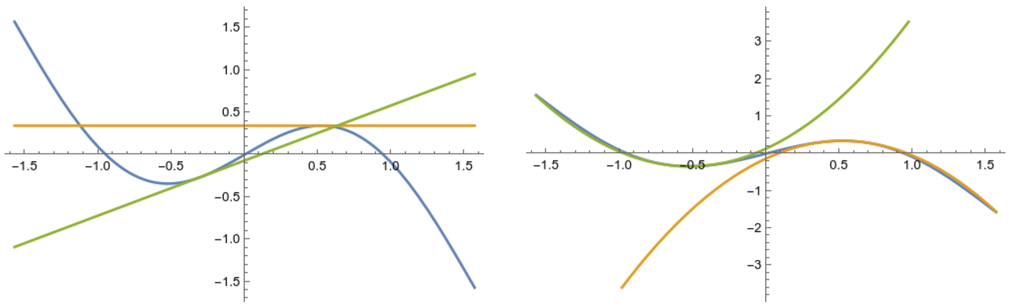


Figure 5.13: The graph of $\sin(2x) - x$. On the left we have first-order Taylor approximations at two points, one of which is a maximum. On the right we have second-order Taylor approximations at two points, which are a maximum and a minimum.

is zero when $x = 0$. (Also at other points, which we'll ignore for now.) But the second derivative is $f''(x) = -2\sin(x^2) - 4x^2\cos(x)$, which is also zero, so the second derivative test doesn't tell us anything. And a graph of $T_2(x, 0)$ indeed gives us a horizontal line.

So let's look at the Taylor series now. We know

$$\begin{aligned}
\cos(x) &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \\
\cos(x^2) &= \sum_{n=0}^{\infty} \frac{x^{4n}}{(2n)!} \\
&= 1 - \frac{x^4}{2} + \frac{x^8}{24} + \cdots \approx 1 - \frac{x^4}{2}.
\end{aligned}$$

So near zero, we can approximate $\cos(x^2)$ with a degree-four polynomial, which opens downwards.

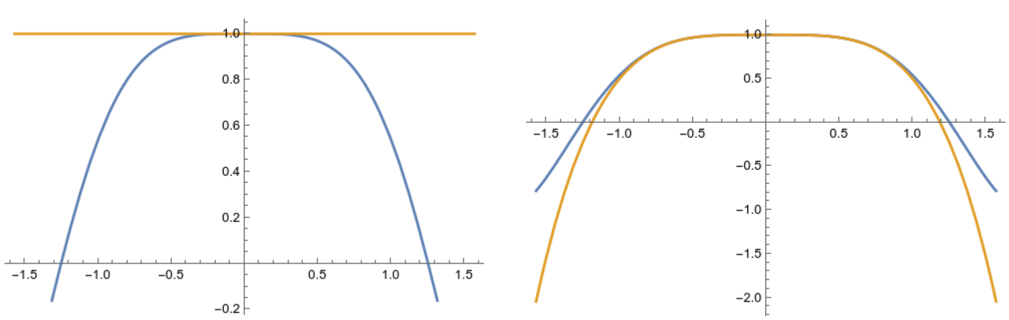


Figure 5.14: The second-order Taylor expansion on the left isn't very helpful, but the fourth-order Taylor expansion shows our function has a maximum at 0..

Example 5.48. But what about the function $g(x) = \sin(x) - x$? Again we'll find that

$g'(0) = g''(0) = 0$, so the second derivative test is useless. But we know

$$\begin{aligned}\sin(x) &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \\ &= x - \frac{x^3}{6} + \frac{x^5}{120} + \dots \\ \sin(x) - x &= -\frac{x^3}{6} + \frac{x^5}{120} + \cdots \approx -\frac{x^3}{6}.\end{aligned}$$

Since $\frac{-x^3}{6}$ doesn't have a maximum or a minimum at 0, $\sin(x) - x$ won't either.

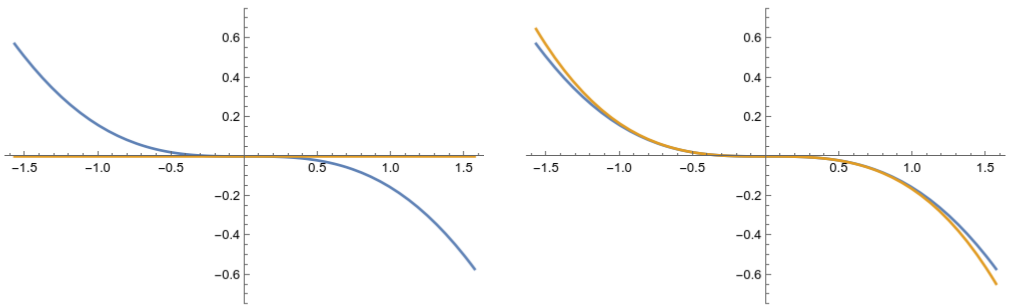


Figure 5.15: The second-order Taylor expansion on the left isn't very helpful, but the third-order Taylor expansion on the right shows that this function has no extrema at 0.

5.5.6 Physical Models

Example 5.49 (Pendulums and Clocks)**.** One place we often use Taylor approximations is in modelling physical systems, such as a pendulum.

We use pendulums in clocks (e.g. grandfather clocks) becaues they keep accurate time. The principle underlying this is the idea that a given pendulum takes the same amount of time to complete a swing regardless of the size of that swing.

This is, unfortunately, false. The angular acceleration on a pendulum (that is, how quickly it changes the angle of rotation) is given by $\alpha = -\frac{mg}{L} \sin\theta$, and thus the position of the pendulum as a function of time satisfies the differential equation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin(\theta),$$

where g is the acceleration due to gravity and L is the length of the pendulum. Solving this differential equation involves a nasty integral that doesn't have a closed-form answer, known as an "elliptic integral." (It's called this because it's also the type of integral used to calculate the circumference of an ellipse).

But you may notice that most clocks have a long pendulum that only makes small arcs. When the angle of the pendulum is small, we can use the Taylor series of sin to approximate $\sin\theta \approx \theta$, and then we have $\alpha \approx -\frac{mg}{L}\theta$, and the approximate differential equation

$$\frac{d^2\theta}{dt^2} \approx -\frac{g}{L}\theta.$$

You might recognize this as the equation $y'' = -ky$ from section 3.3.3, that gave simple harmonic motion. With initial conditions $\theta(0) = \theta_0, \theta'(0) = 0$, we get the solution $\theta(t) \approx \theta_0 \cos\left(\sqrt{g/L}t\right)$, and we see that the time a pendulum takes to complete one swing is $T \approx 2\pi\sqrt{L/g}$, regardless of the initial position. (The error in this approximation causes a typical grandfather clock to lose about 15 seconds a day).

If we need to know the answer to more precision, Taylor series still provide a solution. If we add more terms to our Taylor approximation, we get the formula

$$T = 2\pi\sqrt{L/g}\left(1 + \frac{\theta_0^2}{16} + \frac{11}{3072}\theta_0^4 + \dots\right).$$

Here we see that the period of oscillation does in fact depend on the initial angle θ_0, but if this initial angle is about .1 radians and the pendulum is about a meter then the duration of a swing is about 2 seconds, with an error of roughly a millisecond.

Example 5.50 (Springs in physics)**.** More generally, if you spend more time doing physics, you'll discover that almost every system you want to study is modeled as a collection of springs.

A spring is a system governed by a simple quadratic equation. So any system governed by a quadratic equation can be treated as a spring. So if you take any system and approximate it with the second Taylor polynomial, you get something that looks like a spring.

As examples: we often model light interacting with matter by treating the atom as an electron on a spring. The strength of the chemical bond between two atoms is given by the Lennard-Jones Potential, which is $\left(\frac{R_m}{r}\right)^{12} - 2\left(\frac{R_m}{r}\right)^6$. If we take the Taylor series centered at R_m we get $-1 + 0r + \frac{72}{R_m^2}r^2 + \cdots \approx \frac{72}{R_m^2}r^2 - 1$, which is just a parabola.

As a note: we often refer to this process as "dropping higher order terms"—the constant term is the order-0 term; the linear term is the order-1 term; we have dropped every term of order higher than 2. We sometimes abbreviate even further and call them the H.O.T.

Example 5.51 (Relativity)**.** The last example we want to do concerns special relativity. Relativity includes a number of interesting phenomena that occur when your velocity is relatively large compared to the speed of light. But we know that at low velocities, special relativity should "look like" Newtonian mechanics.

Most of the relativity equations feature a variable γ, where $\gamma = \frac{1}{\sqrt{1-(v/c)^2}}$. We'd like to use the binomial expansion, so we write

$$\gamma = (1-(v/c)^2)^{-1/2} \approx 1 + \frac{-1}{2}(-(v/c)^2) = 1 + \frac{1}{2}\frac{v^2}{c^2}$$

is the first-order Taylor approximation to γ. It should be accurate when v/c is small–that is, when our velocity is very small relative to the speed of light.

Famously, the energy of an object at rest is given by $E = mc^2$. Less famously, the energy of an object in motion is given by $E = mc^2\gamma$; when $v = 0$ then $\gamma = 1$ and we get the famous equation. But what if v is small, but non-zero? We can take the Taylor expansion from before, and get

$$E \approx mc^2\left((1 + \frac{1}{2}\frac{v^2}{c^2}\right) = mc^2 + \frac{1}{2}mv^2.$$

And the second term is just the usual formula for kinetic energy.

Similarly, the formula for time dilation is $T' = T\gamma$. If we take the first-order approximation, we have $T' = T + \frac{T}{2}\frac{v^2}{c^2}$. But even better, if we take the zeroth-order approximation, we have $\gamma \approx 1$ and thus $T' \approx T$. This tells us that at low velocities, time dilation is negligible.

5.6 Bonus Taylor Series Fun

5.6.1 Failure Modes of Taylor Series

Sadly, while Taylor series are awesome, they don't always work. Consider the function defined by $f(x) = e^{-1/x^2}$ and $f(0) = 0$. This function is continuous and in fact differentiable at 0:

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{e^{-1/h^2}}{h} = 0.$$

We can repeat this work, and we see that $f''(0) = f'''(0) = \cdots = f^{(n)}(0) = 0$. Thus the Taylor series is

$$T_f(x,0) = \sum_{n=0}^{\infty}\frac{0}{n!}x^n = 0.$$

But clearly $f(x) \neq 0$ when $x \neq 0$, so f is equal to its Taylor series only in the trivial case when $x = 0$. So just remember: Taylor series *don't always work*.

But once we look at the graph, this makes perfect sense:

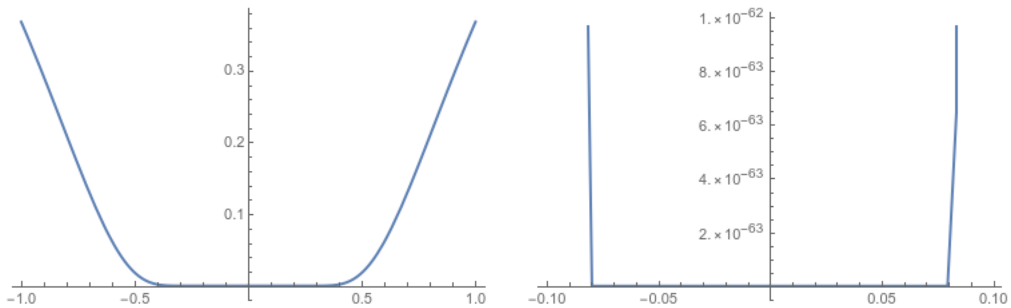


Figure 5.16: The graph of e^{-1/x^2} is absurdly flat near 0.

5.6.2 Power Series and Differential Equations

In section 3.4 we talked about solving separable differential equations, but most differential equations are not separable. There are a lot of tools we can use to solve non-separable equations, but one approach is to use power series—which basically always works.

Example 5.52. Recall the classic differential equation $y' = y$. Suppose the function $y(x)$ can be represented by a power series. Then we have

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \dots \\ y' &= a_1 + 2a_2 x + 3a_3 x^2 + \dots \end{aligned}$$

and since these are the same function, each coefficient has to match up. So we get the system

$$\begin{aligned} a_1 &= a_0 & \qquad a_1 &= a_0 \\ 2a_2 &= a_1 & \qquad a_2 &= a_1/2 \\ 3a_3 &= a_2 & \qquad a_3 &= a_2/3 \\ &\vdots & &\vdots \end{aligned}$$

So if we know a_0, then we can figure out all the other coefficients.

How do we find a_0? Well, that's a choice. Remember any differential equation will wind up with free constants in the end. But if we take $a_0 = 1$, which seems like a reasonable choice, we then get

$$\begin{aligned}
a_1 &= a_0 = 1 \\
a_2 &= a_1/2 = 1/2 \\
a_3 &= a_2/3 = 1/6 \\
&\vdots \\
a_n &= a_{n-1}/n = 1/n! \\
y &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots
\end{aligned}$$

which is exactly the Taylor series for e^x we worked out in section 5.3.

(What happens if we choose a different a_0? That just multiplies all the coefficients by a constant, so if $a_0 = C$ then our power series gives us Ce^x. And we already know that Ce^x is the solution to this differential equation!)

Example 5.53. Solve $y'' - 3xy' + y = 0$.

We don't have any tools for this, so we use Taylor series. Assume $y = \sum_{n=0}^\infty c_n x^n$, and then we have

$$\begin{aligned}
\left(\sum_{n=0}^\infty c_n x^n\right)'' - 3x\left(\sum_{n=0}^\infty c_n x^n\right)' + \sum_{n=0}^\infty c_n x^n &= 0 \\
\sum_{n=0}^\infty c_n n(n-1)x^{n-2} - 3x\sum_{n=0}^\infty c_n n x^{n-1} + \sum_{n=0}^\infty c_n x^n &= 0 \\
\sum_{n=0}^\infty c_{n+2}(n+1)(n+2)x^n - 3\sum_{n=0}^\infty c_n n x^n + \sum_{n=0}^\infty c_n x^n &= 0 \\
\sum_{n=0}^\infty \left((n+1)(n+2)c_{n+2} - 3nc_n + c_n\right)x^n &= 0
\end{aligned}$$

and thus for each n we have

$$\begin{aligned}
(n+1)(n+2)c_{n+2} &= (3n-1)c_n \\
c_{n+2} &= \frac{(3n-1)c_n}{(n+1)(n+2)}
\end{aligned}$$

as our recurrence relation. As before, we see that our solution must have the form

$$\begin{aligned}
y &= \sum_{k=0}^\infty \frac{c_0((5)(11)\cdots(6k-1))}{(2k)!}x^{2k} + \frac{c_1((8)(14)\cdots(6k+2)}{(2k+1)!}x^{2k+1} \\
&= c_0\sum_{k=0}^\infty \frac{(5)(11)\cdots(6k-1)}{(2k)!}x^{2k} + c_1\sum_{k=0}^\infty \frac{(8)(14)\cdots(6k+2)}{(2k+1)!}x^{2k+1}.
\end{aligned}$$

Example 5.54. In section 5.1 I mentioned the *Bessel function*

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

This arises naturally as a solution to the differential equation

$$x^2y'' + xy' + x^2y = 0$$

which is used to study a lot of physics on cylinders.

5.6.3 Taylor Series and Complex Numbers

Perhaps the most surprising and important fact about the trigonometric power series is the way they combine. You'll notice that for both sin and cos, every term looks like $\frac{x^n}{n!}$ but neither series has all the terms.

Leonhard Euler, around 1740, asked himself what it means to exponentiate an imaginary number. Since the Taylor series of e^x agrees with the function everywhere on the real line, it makes sense to define $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. If we plug in a purely imaginary number ix for $x \in \mathbb{R}$, we see:

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^{2n}x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i(i)^{2n}x^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \cos(x) + i\sin(x). \end{aligned}$$

Thus we obtain *Euler's Formula*:

$$e^{i\theta} = cos\theta + i\sin\theta.$$

As a corollary, we get the statement Euler called the most beautiful in all mathematics: $e^{i\pi} = \cos(\pi) + i\sin(\pi) = 1$, or

$$e^{i\pi} - 1 = 0.$$

This statement relates the five most fundamental constants in the complex plane.

But in addition to being really pretty, this also has a geometric interpretation. If we think about the point $\cos\theta + i\sin\theta$ on the complex plane, it has x-coordinate $\cos\theta$ and y-coordinate $\sin\theta$ and is thus the point on the unit circle corresponding to angle θ. (This is why the unit circle is oriented as it is!)

So in general we can represent any point on the unit circle as $e^{i\theta}$, where θ is the angle the point makes from the positive x-axis. Further, if we have *any* complex number z, we can represent it in "polar coordinates" by giving its absolute value $|z|$ and its complex argument θ. Thus for any complex number z, we have

$$z = |z|e^{i\arg(z)}.$$

Another result from this line of thinking is De Moivre's Formula:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

There are some functions, unfortunately, that don't behave nearly so nicely, such as the logarithm. Unlike exp and sin and cos we can't just extend the power series for log to the whole complex plane, since it has finite radius of convergence (and an unavoidable pole at 0).

But in fact the problem is deeper. We generally want to define log to be the inverse of exp, so that $\log\exp z = z$ for every z. But notice that $\exp(0) = 1$, and also $\exp(2\pi i) = 1$. So if we want to define log on the whole complex plane we must have $\log 1 = 0$ and also $\log 1 = 2\pi i$ and this is obviously a problem, since functions can't have multiple outputs. We in fact have infinitely many numbers z with $e^z = 1$, or in fact any complex number you choose (except 0; $\log 0$ is never defined).

We solve this by choosing a "branch," which basically corresponds to which complex arguments we allow; we will typically require our arguments to be in $(-\pi, \pi]$; this is called the "principal value" of the argument. (Notice this is similar to the way we require $\arcsin x$ to be in $[-\pi, \pi]$ in section 1.5). Notice also that in this case, we aren't really happy at negative real numbers–there's a huge jump discontinuity there in the complex argument.

There is another option, which avoids this discontinuity, and which I mention mainly because it's cool. We can define what's called a *Riemann Surface*, which is a two-dimensional surface that we can think of as sitting in three-dimensional space. In this case our Riemann surface looks like a giant helical Archimedes Screw.

This surface "covers" the complex plane with infinitely many "sheets." The logarithm is a function defined on this surface; which sheet we are in tells us which "branch" the argument should be in, and other than that the logarithm is defined as it would be for the point of the complex plane "under" our surface. Thus the logarithm of a point on one sheet would be 0, and the logarithm of the point one sheet above it is $2\pi i$, and if we go up another sheet we get $4\pi i$, and so on. See figure 5.17 for a sketch of what this might look like.

This behavior also occurs with functions like $\sqrt[n]{z}$. Since every (non-zero) number has two square roots, the square root function is "doubly ramified" or a "two-fold cover" of

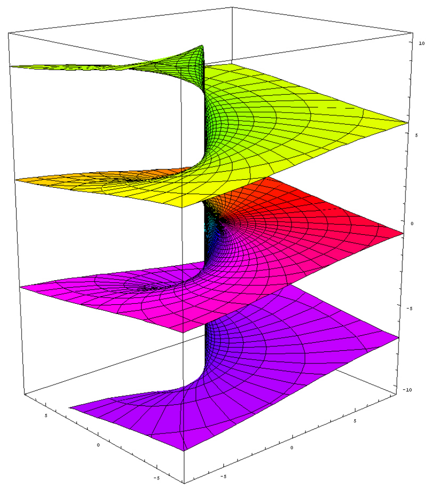


Figure 5.17: The Riemann surface corresponding to the complex logarithm

the complex plane. The nth-root function is an n-fold cover. They are, in fact, essentially the same picture as the logarithm picture, but with only finitely many sheets, which wrap around and join up. By convention, we put the discontinuity still on the negative real line. We can see an attempted picture in figure 5.18.

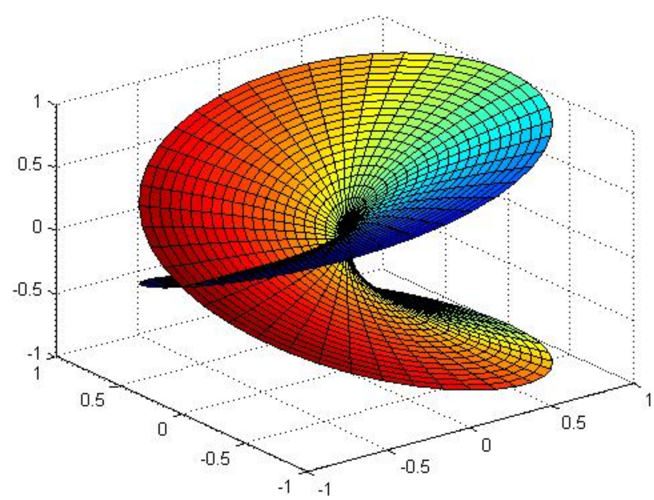


Figure 5.18: The Riemann surface corresponding to the square root function

Special attention should be paid to the idea of an “nth root of unity”, where “unity” is just a fancy word for “one.” We know that a number has two square roots; for instance the number one has 1 and -1 as square roots. By the same logic, any number should have n distinct nth roots.

A little creativity shows that the nth roots of unity are the numbers are $e^{2\pi ik/n}$, since

$$\left(e^{2\pi ik/n}\right)^n = e^{2\pi ik} = \left(e^{2\pi i}\right)^k = 1^k = 1.$$

These are points spaced evenly around the unit circle. They are very useful in creating functions and other operations that have certain types of "periodicity", which means they repeat every n times. (The most recognizable periodic function is $\sin(x)$, which has a period of 2 pi.)

The roots of unity are especially useful in conjunction with Fourier series, which give a useful way of representing a periodic function as an infinite sum of sine and cosine functions. They are an alternative tool to Taylor series that are useful in situations where Taylor series don't work terribly well.

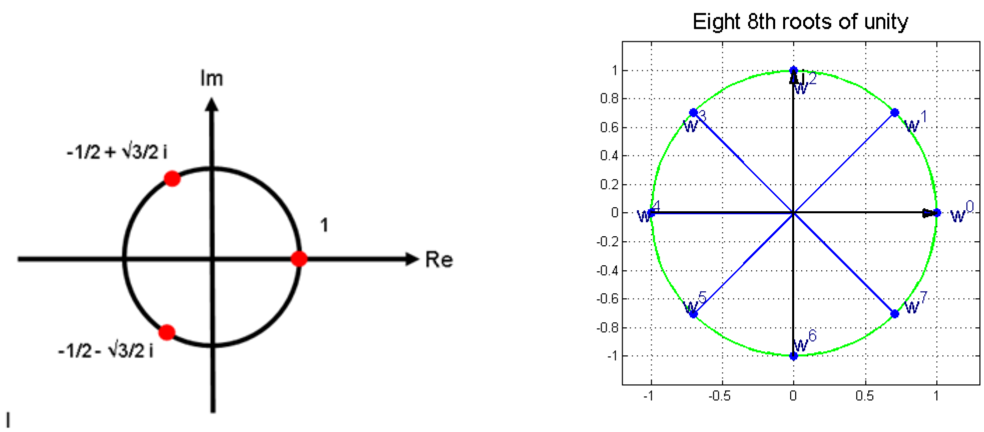


Figure 5.19: Cube Roots and Eighth Roots of Unity

5.7 Double Bonus: Real Fourier Series

We've seen that we can represent many functions as a power series, an infinite sum of multiples of powers of x. But some functions are hard to represent this way, and we want other tools. In particular we can represent a function as an infinite sum of trigonometric functions.

For this discussion we'll confine ourselves to real functions on the interval $[-\pi, \pi]$. (The same idea works for functions on any closed interval, but it's easier to talk about just this particular interval for right now. Also, π has popped up again for no reason. Hi, π!)

Definition 5.55. Let f be a function on the interval $[-\pi, \pi)$. Then the *Fourier series* of f

is given by

$$f_\infty(x) = \frac{C}{2} + \sum_{n=1}^{\infty} \left(a_n \sin(nx) + b_n \cos(nx)\right)$$

where

$$\begin{aligned}
C &= 2\langle f(x), 1\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
a_n &= 2\langle f(x), \sin(nx)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\
b_n &= 2\langle f(x), \cos(nx)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx.
\end{aligned}$$

For Taylor series we could match things up by taking derivatives; to find Fourier series coefficients we instead compute integrals. That makes the computations much nastier! But the computations are *doable*, and they make sense because the integral of one term times a different term is always zero.

Before we dive into computations to prove this, let's think about why we should expect it to be true. A sin or cos function passes through a complete cycle between $-\pi$ an π, so the positive bits will exactly cancel out the negative bits. When we multiply two different sin or cos functions, they don't correlate with each other—each one passes through cycles at a different rate from the others, so the cycles don't reinforce or cancel out. Thus we'll still have exactly as much on top as we do on bottom, and the integrals should be zero.

Proof. We use the notation $\langle f, g\rangle$ to represent $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$.

$$\begin{aligned}
2\pi\langle 1, \sin(nx)\rangle &= \int_{-\pi}^{\pi} \sin(nx)\,dx = \left.\frac{-\cos(nx)}{n}\right|_{-\pi}^{\pi} = \frac{1}{n} - \frac{1}{n} = 0. \\
\langle 1, \cos(nx)\rangle &= \int_{-\pi}^{\pi} \cos(nx)\,dx = \left.\frac{\sin(nx)}{n}\right|_{-\pi}^{\pi} = 0 - 0 = 0.
\end{aligned}$$

The products of the sin and cos functions are a bit trickier.

$$\begin{aligned}
\langle \sin(nx), \sin(mx)\rangle &= \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx \\
&= -\frac{\cos(nx)}{n}\sin(mx)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{m}{n}\cos(nx)\cos(mx)\,dx \\
&= 0 + \frac{m}{n^2}\sin(nx)\cos(mx)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{m^2}{n^2}\sin(nx)\sin(mx)\,dx \\
0 &= \left(\frac{m^2}{n^2} - 1\right)\int_{-\pi}^{\pi} \sin(nx)\sin(mx)
\end{aligned}$$

As long as $\frac{m^2}{n^2} \neq 1$ this implies that $\langle \sin(nx) \sin(mx) \rangle = 0$. For positive integers m, n, this holds whenever $m \neq n$. In contrast

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin^2(nx)\,dx &= \int_{-\pi}^{\pi} \frac{1-\cos(2x)}{2}\,dx \\
&= \left(\frac{x}{2} - \frac{\sin(2nx)}{4}\right)\Big|_{-\pi}^{\pi} = \left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = \pi. \\
\langle \sin(nx), \sin(nx) \rangle &= \frac{1}{2\pi} \cdot \pi = \frac{1}{2}.
\end{aligned}$$

Similar arguments work for $\langle \cos(nx), \sin(mx) \rangle$ and $\langle \cos(nx), \cos(mx) \rangle$. □

Theorem 5.56. *Suppose f is a continous function with continuous derivative, except for finitely many points, on $[-\pi, \pi)$. Then $f(x)$ is equal to its Fourier series except for at finitely many points.*

Notice that unlike in the case of Taylor series, this always works. every *continuous function is (essentially) equal to its Fourier series.*

What does this mean? It means that if we have a function on $[-\pi, \pi)$ then we can look at it as being composed of a bunch of different "waves" of different frequencies, and the coefficients tell us how large each wave is. (The constant term tells us the average value around which the waves are oscillating). Further, a Fourier series is always a periodic function on the whole real line. So any periodic function can be viewed as a Fourier series, and this technology allows us to see it as composed of many smaller simpler waves. We'll return to the physics and geometry of this soon.

Example 5.57. Let $f(x) : [-\pi, \pi) \to \mathbb{R}$ be given by $f(x) = x$. The periodic version of this function is a "sawtooth wave." Then we have:

$$\begin{aligned}
\frac{C}{2} &= \langle f(x), 1 \rangle \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} x dx \\
&= \frac{1}{2\pi} x^2 |_{-\pi}^{\pi} = 0.
\end{aligned}$$

$$\begin{aligned}
a_n &= 2\langle f(x), \sin(nx)\rangle \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx \\
&= \frac{1}{\pi}\left(-x\cdot\frac{\cos nx}{n}|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos(nx)}{n}dx\right) \\
&= \frac{1}{\pi}\left(\frac{-\pi\cos(n\pi) - \pi\cos(-n\pi)}{n}\right) \\
&= -2\frac{\cos(n\pi)}{n} = (-1)^{n+1}\frac{2}{n}.
\end{aligned}$$

$$\begin{aligned}
b_n &= 2\langle f(x), \cos(nx)\rangle \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)dx \\
&= \frac{1}{\pi}\left(x\cdot\frac{\sin nx}{n}|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin(nx)}{n}dx\right) \\
&= \frac{1}{\pi}\left(\frac{\pi\sin(n\pi) + \pi\sin(-n\pi)}{n}\right) \\
&= 0.
\end{aligned}$$

Thus the sawtooth wave has Fourier series

$$f(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nx).$$

Example 5.58. Define sgn : $[-\pi, \pi] \to \mathbb{R}$ by $f(x) = -1$ if $x < 0$ and $f(x) = 1$ if $x \geq 0$. (Made periodic, this is a "square wave").

$$\begin{aligned}
\frac{C}{2} &= \langle \operatorname{sgn}(x), 1/2\rangle \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\operatorname{sgn}(x)dx = 0.
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{2\pi}\langle \operatorname{sgn}(x), \sin(nx)\rangle \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} -\sin(nx)dx + \int_0^{\pi} \sin(nx)dx\right) \\
&= \frac{1}{\pi}\left(\frac{\cos(nx)}{n}|_{-\pi}^{0} - \frac{\cos(nx)}{n}|_0^{\pi}\right) \\
&= \frac{1}{n\pi}(\cos(0) - \cos(-n\pi) - \cos(n\pi) + \cos(0) \\
&= \frac{2}{n\pi}(\cos(0) - \cos(n\pi))
\end{aligned}$$

which equals $\frac{r}{n\pi}$ if n is odd and 0 if n is even.

$$\begin{aligned}
b_n &= \frac{2}{2\pi}\langle \operatorname{sgn}(x), \cos(nx)\rangle \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} -\cos(nx)dx + \int_0^{\pi} \cos(nx)dx\right) \\
&= \frac{1}{\pi}\left(\frac{-\sin(nx)}{n}|_{-\pi}^{0} + \frac{\sin(nx)}{n}|_0^{\pi}\right) \\
&= \frac{1}{\pi n}(-\sin(0) + \sin(-n\pi) + \sin(n\pi) - \sin(0)) = 0.
\end{aligned}$$

Thus

$$\operatorname{sgn}(x) = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin((2n+1)x).$$

6 Parametrization and Coordinates

In this section we're going to dabble a bit in working with multiple variables. We'll specifically going to look at the way we can describe some plane geometry ideas that we haven't thought much about in the past. You can view this as a teaser for multivariable calculus; but it also gives you a couple of basic ideas that are useful to have in your pocket if you don't actually take multi.

6.1 Curves and Motion

In this section we want to study curves in the plane. By a curve we mean, essentially, any shape that is in some sense "one-dimensional". So a line, a circle, and a curving spiral space are all curves.

The essence of a curve is the one-dimensionality. We capture this idea by requiring position on our curves to be described by one single real number. That is, we can describe our position on the curve with exactly one coordinate. We say a system of coordinates for an object is a "parametrization", because it describes the object with some number of parameters.

Definition 6.1. We say a pair of equations $x = f(t), y = g(t)$ is a *parametrization of a curve* in the plane.

Sometimes you'll see these functions just called $x(t)$ and $y(t)$. It's common to refer to the pair with its own name: we can write $\vec{r}(t) = (f(t), g(t))$.

Example 6.2. Let's find a parametrization for the curve $y = x^2$.

We see that we can parametrize this by the function $x = t, y = t^2$. You'll notice that this is basically the original function formula: we have $x = t$ and $y = t^2 = x^2$. Any time we have a curve that is the graph of a function, we effectively have a parametrization for free; the input variable gives us a parametrization.

Example 6.3. Let's parametrize a circle of radius 1. Notice that we can't use the same trick as last time, since this isn't a function.

We could try something like $f(t) = t, g(t) = \sqrt{1 - t^2}$ for $-1 \leq t \leq 1$. This sort of works, but only captures the top half of the circle. We could keep trying to make this idea work, but it basically won't.

Instead, we take advantage of the fact that circles are fundamentally trigonometric. We see that $x = \cos(t), y = \sin(t)$ will give us every point on the circle—in fact, this is the usual unit circle definition of sin and cos. In particular, we have $(f(0), g(0)) = (1, 0)$ is the rightmost point of the circle, and as t increases we move counterclockwise around the circle.

However, this isn't the only possible parametrization. For instance, we could instead take $x = \sin(t), y = \cos(t)$. This will still parametrize the circle, but it starts at $(x, y) = (0, 1)$ which is the top of the circle, and proceeds clockwise. So we get the same *shape* but a different *path*.

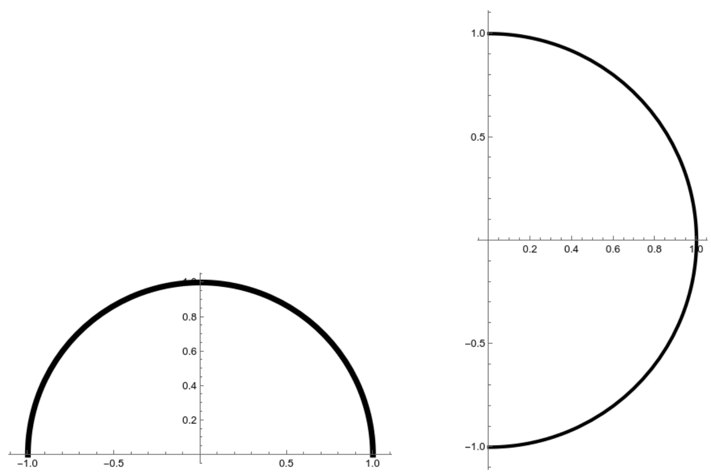


Figure 6.1: The graphs of $(\cos(t), \sin(t))$ and $(\sin(t), \cos(t))$ for $0 \leq t \leq \pi$.

Example 6.4. Another nice property of parametrizations is that it's easy to shift them in space. Let's parametrize a circle of radius 2 centered at $(3, 2)$, going counterclockwise starting from the right-hand point.

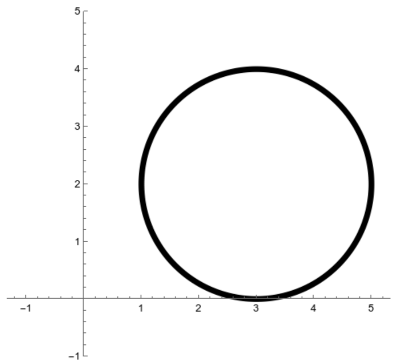
We know that a circle of radius 1 centered at the origin is $\vec{r}(t) = (\cos(t), \sin(t))$. To get radius 3, we multiply by 3; then to shift the center, we add $(3, 2)$, leaving us with the parametrization $\vec{r}(t) = (3 + 2\cos(t), 2 + 2\sin(t))$.

If we want to start from left-hand point and go clockwise, we can do a couple things. One is to flip the circle upside down and start halfway around; this would give $\vec{r}(t) = (3 + 2\cos(t + \pi), 2 - 2\sin(t + \pi))$.

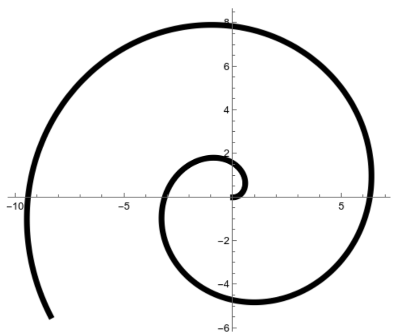
Alternatively, we could start from the parametrization $(\sin(t), \cos(t))$, which already goes clockwise. Then we would get $\vec{r}(t) = (3 + 2\sin(t - \pi/2), 2 + 2\cos(t - \pi/2))$.

In general, choices of parametrization aren't unique. Often we can make a problem easier (or harder) by changing our choice of coordinates.

Example 6.5. Let's consider the curve given by $x = 5\cos t, y = 5\sin t$. This gives us a circle of radius 5.



We can make this more interesting by taking something like $x = t\cos(t), y = t\sin(t)$. This will create a shape that spirals outwards.



We can also make more fun shapes with parametrization.

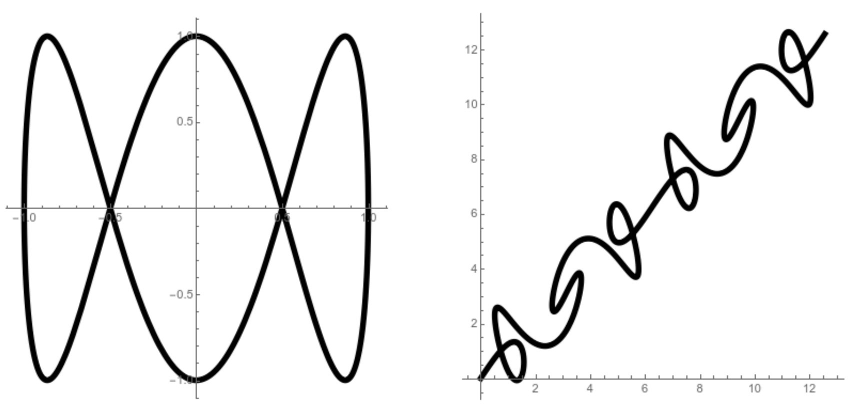


Figure 6.2: $\vec{r}(t) = (\cos(t), \sin(3t))$ and $\vec{s}(t) = (t + \sin(3t), t + \sin(5t))$

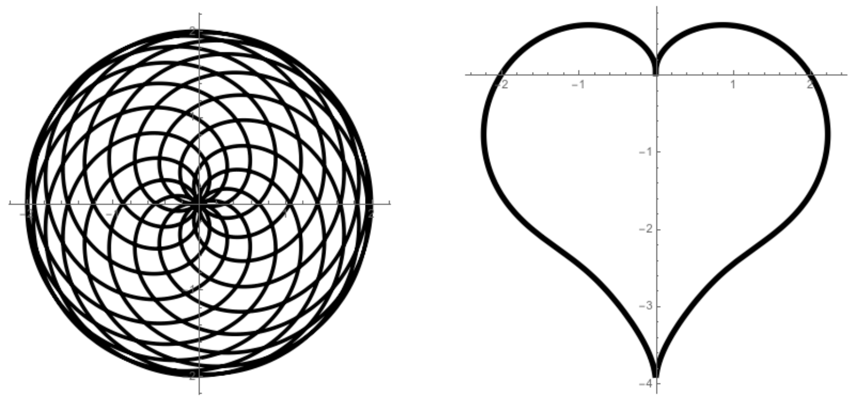


Figure 6.3: $\vec{r}(t) = (\cos(10t) + \cos(21t), \sin(10t) + \sin(21t))$ and
$$\vec{s}(t) = \left(\cos(t)\left(\frac{\sin(t)\sqrt{|\cos(t)|}}{\sin(t)+7/5} - 2\sin(t) + 2\right), \sin(t)\left(\frac{\sin(t)\sqrt{|\cos(t)|}}{\sin(t)+7/5} - 2\sin(t) + 2\right)\right)$$

I'm always surprised that complicated-looking shapes sometimes have very simple parametrizations, whereas simple shapes like the heart have sometimes very complicated curves. This is closely related to how much "like a circle" the shape is. Some shapes fall naturally out of throwing together elementary functions, and others do not.

It turns out that for any curve, it's possible to find a parametrization using the magic of *Fourier series*. But the formulas tend to wind up looking pretty ridiculous.

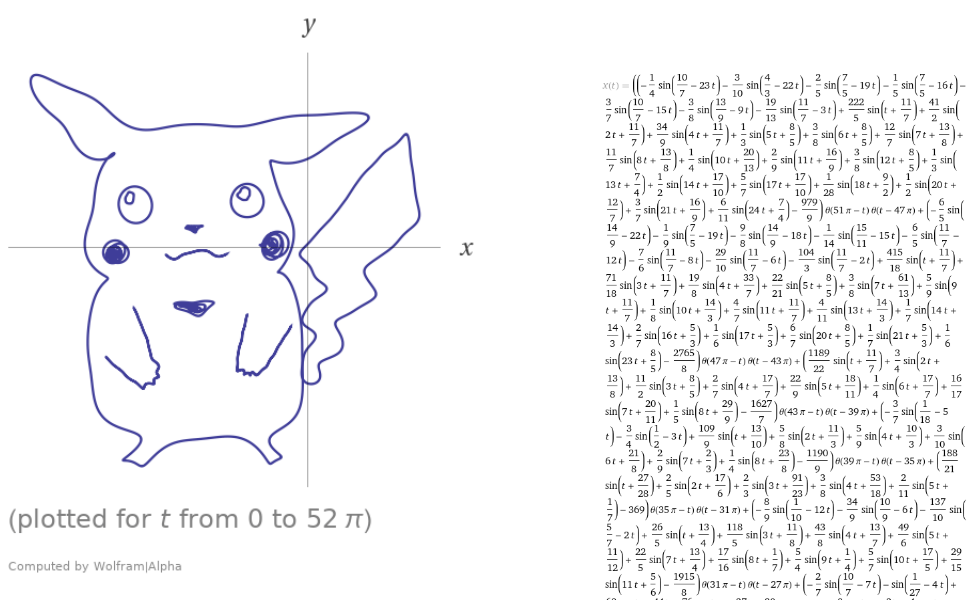


Figure 6.4: See more examples at `https://www.wolframalpha.com/examples/mathematics/geometry/curves-and-surfaces/popular-curves/`

6.1.1 Calculus of Curves

So far we've discussed parametric equations as giving position as a function of time, and talking about the direction and sometimes the speed of motion. As in the single-variable case, we can make this more precise by the theory of derivatives.

Speed is change in position with respect to time. We can define this pretty easily:

Definition 6.6. The *velocity* of an object that moves along a path with position $\vec{r}(t)$ at time t is

$$\vec{v}(t) = \vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

What is this really doing? We're just defining $\vec{r}\,'(t)$ to be the amount that $\vec{r}(t)$ changes by if we increase t by a little bit, which is just how the normal derivative works. Notice that $\vec{r}\,'(t)$ has two pieces, the x component and the y component. These represent the amount of change in the x coordinate and the y coordinate, respectively. Together we call them the *tangent vector* to $\vec{r}(t)$ at a point. We can define a *parametric tangent line* by looking at the function

$$\vec{T}(t) = \vec{r}(a) + \vec{r}\,'(a)(t-a).$$

The

Just like the definition of a normal derivative, this definition is a bit hard to work with. In the single variable case we came up with a bunch of rules we could use to compute derivatives. Here we don't need to go through all of that again: the x and y coordinates change independently, so we can consider them independently. (This is implicitly because derivatives are always linear, so we can write the derivative of a sum as the sum of the derivatives).

We really just have two single-variable derivatives. And we already know how to compute those.

Proposition 6.7. *Let $\vec{r}(t) = (f(t), g(t))$ be a parametrization. Then*

$$\vec{r}\,'(t) = (f'(t), g'(t)).$$

Proof.

$$\begin{aligned}
\vec{r}\,'(t) &= \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \\
&= \lim_{h\to 0} \frac{(f(t+h), g(t+h)) - (f(t), g(t))}{h} \\
&= \lim_{h\to 0} \left(\frac{f(t+h) - f(t)}{h}, \frac{g(t+h) - g(t)}{h} \right) \\
&= (f'(t), g'(t)).
\end{aligned}$$

□

Example 6.8. Consider the circle parametrized by $(\cos(t), \sin(t))$. Then the derivative is $\vec{r}\,'(t) = (-\sin(t), \cos(t))$.

If we want to find the tangent vector at the point $(1, 0)$, we compute the derivative and plug in $t = 0$, so we get $\vec{r}\,'(0) = (0, 1)$ as your vector, and the tangent line is

$$(1, 0) + (0, 1)(t - 0) = (1, 0 + t).$$

Now suppose want the tangent line at $(\sqrt{2}/2, \sqrt{2}/2)$. This occurs at time $t = \pi/4$ and so we compute $\vec{r}\,'(\pi/4) = (-\sqrt{2}/2, \sqrt{2}/2)$. Thus the tangent line is

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) + \left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)(t - \pi/4) = \sqrt{2}/2(1 - t, 1 + t).$$

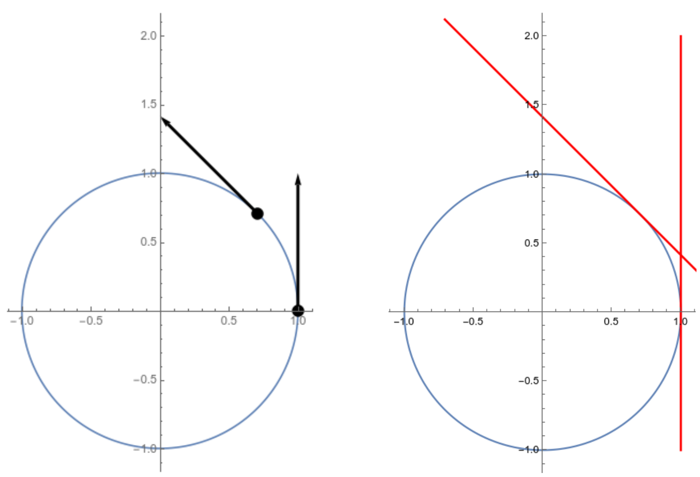


Figure 6.5: Left: the tangent vectors. Right: the tangent lines.

After taking the first derivative, we can also take the second (and further) derivatives. As in the single variable case, if the function gives position, and the derivative gives velocity, then the second derivative gives acceleration.

Definition 6.9. The *acceleration* of an object that moves along a path with position $\vec{r}(t)$ at time t is

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = \frac{d^2\vec{r}}{dt^2} = \lim_{h \to 0} \frac{\vec{r}\,'(t+h) - \vec{r}\,'(t)}{h}.$$

As you'd expect, we can compute the acceleration just by taking the componentwise second derivatives: we have

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = (f''(t), g''(t)).$$

Example 6.10. Consider again the circle parametrized by $\vec{r}(t) = (\cos(t), \sin(t))$. Then we know that $\vec{r}\,'(t) = (-\sin(t), \cos(t))$, and thus the second derivative is $\vec{r}\,''(t) = (-\cos(t), -\sin(t))$.

Then we compute that $\vec{r}\,''(0) = (-1, 0)$ and $\vec{r}\,''(\pi/4) = (-\sqrt{2}/2, -\sqrt{2}/2)$.

We notice (figure 6.6 that the acceleration arrows in a circle always point inwards! This is because the motion is at a constant speed, and so the acceleration is only changing direction; so we can't speed up or slow down in the direction of our velocity, and our acceleration must be perpendicular to our velocity.

If we want, we can use the second (and further) derivatives to build an analogue of the Taylor series. For instance, using the second derivative we can get parabolic approximations to our circle. We wind up with the parabolas of figure 6.6

$$(1,0) + (0,1)(t-0) + (-1,0)\frac{1}{2}(t-0)^2 = \left((1 - \frac{t^2}{2}, t\right)$$

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) + \left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)(t - \pi/4) + \left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)\frac{1}{2!}(t-\pi/4)^2$$
$$= \sqrt{2}/2\left(1 - t - \frac{t^2}{2}, 1 + t - \frac{t^2}{2}\right).$$

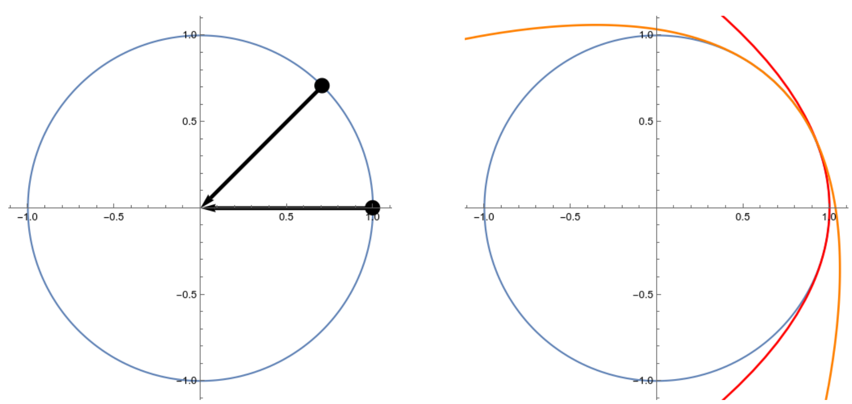


Figure 6.6: Left: the acceleration vectors. Right: the second-order approximations

However, these parametrizations have subtleties that aren't just their shapes.

Example 6.11. Suppose we have the function $\vec{r}(t) = (2 + 4t^3 + 4t, 6 + 3t^3 + 3t)$. Then we can compute the velocity to be $\vec{r}'(t) = (12t^2 + 4, 9t^2 + 3)$, and the acceleration is given by $\vec{r}''(t) = (24t, 18t)$.

But if we graph the function, it just looks like a line! We have $r(t) = (2, 6) + (t^3 + t)(4, 3)$, so we're always on the line with slope $3/4$. Then why is the velocity so variable? We're constantly maintaining the same *direction*, but the *speed* in which we move in that direction changes, as we see in figure 6.7.

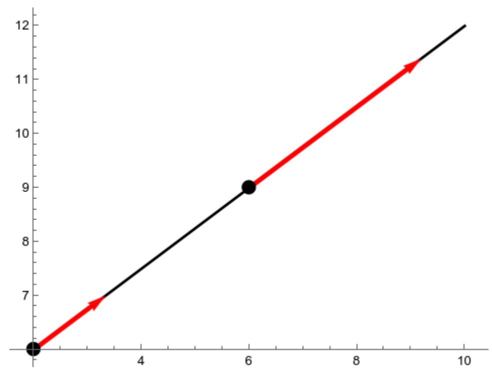


Figure 6.7: (Scaled) velocity vectors starting at $(2, 6)$ and at $(6, 9)$

We can also compute the arc length for parametrized curves just like we can for regular curves. Remember in section 3.2.1 we saw that the arc length of a curve was

$$\begin{aligned} L &= \int_{a_1}^{b_1} ds \\ &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_c^d \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy \end{aligned}$$

(where the bounds of the integral depend on the variable we're integrating with respect to, so x goes from a to b as y goes from c to d).

Now suppose we have a parametrized curve $(x, y) = (f(t), g(t))$. We can take α, β so that $f(\alpha) = a$ and $f(\beta) = b$, so that as t goes from α to β we have x going from a to b. Then a change of variables gives

$$\begin{aligned} L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} \frac{dx}{dt}\, dt \\ &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \end{aligned}$$

This works even if our curve isn't given by writing y as a function of x; we have the same basic argument that an infinitesimal line segment has length roughly

$$\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \Delta t.$$

Then we can add up the lengths of each short line segment, and take the limit as the number of segments goes to zero.

Example 6.12. We can parametrize a circle by $(x, y) = (\cos(t), \sin(t))$ as t varies from 0 to 2π. Then we have

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2}\, dt \\ &= \int_0^{2\pi} 1\, dt = 2\pi. \end{aligned}$$

What if we instead let t vary from 0 to 4π? We get

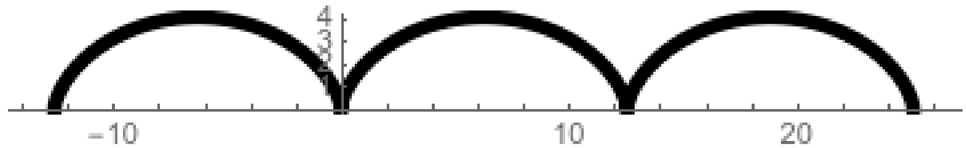
$$\begin{aligned} L &= \int_0^{4\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2}\, dt \\ &= \int_0^{4\pi} 1\, dt = 4\pi. \end{aligned}$$

But the apparent curve hasn't changed! We get twice as much arc length because we're traveling around the circle twice; this calculation finds the distance actually covered by the parametrization, even if you're repeating the same path multiple times.

Finally, we can find the area under a parametric curve, in a kind of silly way. We know that area is given by the integral $A = \int_a^b y\, dx$. If we have a parametrization $y = g(t)$ and $x = f(t)$, then we can do a u-substitution $x = f(t)$. Then $dx = f'(t)\, dt$ and we get

$$A = \int_a^b y\, dx = \int_a^b g(t) f'(t)\, dt.$$

Example 6.13. Consider the curve $(x, y) = \big(2\big(t - \sin(t)\big), 2\big(1 - \cos(t)\big)\big)$.



If we want the area under one arch, we have

$$\begin{aligned}\int_0^{4\pi} y\,dx &= \int_0^{2\pi} 2(1 - \cos(t)) \cdot \frac{d}{dt} 2(t - \sin(t))\,dt \\ &= \int_0^{2\pi} 4(1 - \cos^2(t))\,dt = 12\pi.\end{aligned}$$

6.2 Polar Coordinates

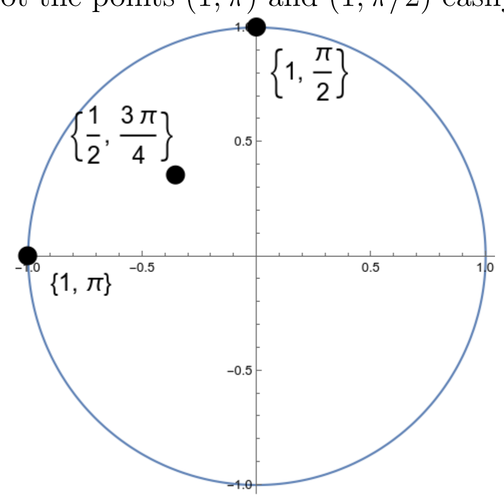
We usually work in a coordinate system known as *rectilinear* or *Cartesian* coordinates (after the French mathematician and philosopher René Descartes, who invented the idea of describing shapes with equations). These coordinates are often very useful, but sometimes they're not the best choice; for instance, they handle circles a bit awkwardly. It's much easier to describe a circle or circle-like region in a new coordinate system based on circular motion.

Definition 6.14. The *polar coordinates* of a point P in the plane are a pair of numbers (r, θ), where r is the distance between P and the origin O, and θ is the angle between the positive x-axis and a line from the origin to P.

We usually choose these numbers so that r is positive, and $\theta \in [0, 2\pi)$.

(Your textbook allow negative r coordinates, which is important for dealing with certain types of equations, but usually isn't very helpful.)

Example 6.15. We can plot the points $(1, \pi)$ and $(1, \pi/2)$ easily. What about $(1/2, 3\pi/4)$?



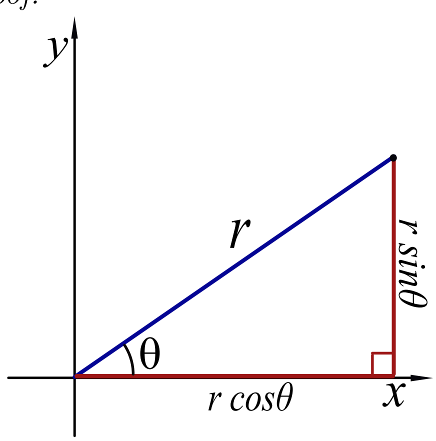
We see that the same point can have many different names in polar coordinates; in fact, $(r, \theta) = (r, \theta + 2\pi)$ for any point (r, θ). We generally either choose to have θ in $[-\pi, \pi]$, or in $[0, 2\pi)$. But if it's convenient you can choose it to have any size.

It's useful to be able to convert between polar coordinates and cartesian coordinates. But some simple trigonometry makes that easy.

Proposition 6.16. *Suppose (x, y) are the cartesian coordinates of a point P, and $(r\,theta)$ are the polar coordinates. Then:*

- $x = r\cos\theta$
- $y = r\sin\theta$
- $r = \sqrt{x^2 + y^2}$
- $\theta = \pm\arctan y/x$.

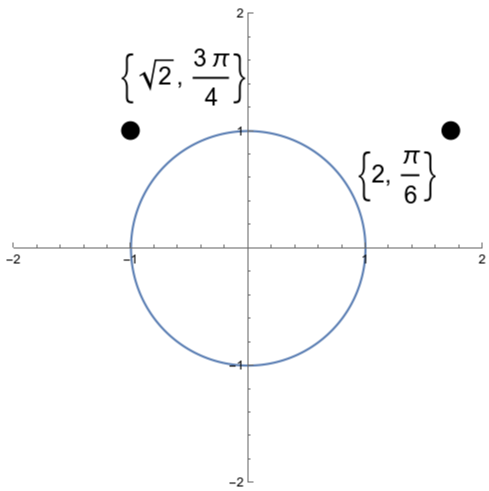
Proof.



□

Example 6.17. If we convert $(2, \pi/6)$ to Cartesian coordinates, we get $(\sqrt{3}, 1)$.

If we convert $(-1, 1)$ to polar coordinates, we get $(\sqrt{2}, 3\pi/4)$. Alternatively, we could say $(\sqrt{2}, -5\pi/4)$ or $(\sqrt{2}, 11\pi/4)$. Polar coordinates have multiple labels for the same point.



We can also graph curves in polar coordinates.

Example 6.18. Circular shapes tend to be easier to describe in polar coordinates.

- The polar equation for a circle of radius c is as simple as possible: it's just $r = c$. The closed disk of radius c is given by the set $\{(r, c) : 0 \leq r \leq c, 0 \leq \theta < 2\pi\}$. The Cartesian coordinates are $\{(x, y) : x^2 + y^2 \leq c^2\}$.
- What does the equation $\theta = 1$ look like? It's a line starting at the origin and going up and to the right at a $\pi/4$ or $45°$ angle. (If we allow r to be negative, the line extends in both directions.)

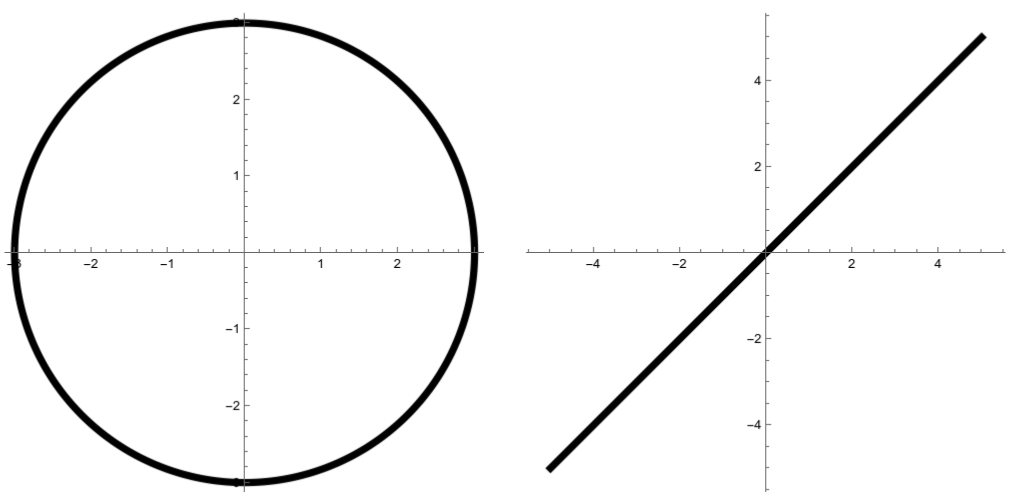


Figure 6.8: Left: The graph of $r = 3$. Right: the graph of $\theta = 1$.

- The wedge of the closed unit disk in the first (upper-right) quadrant is $\{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq \pi/2\}$. The Cartesian coordinates are $\{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 1.\}$

- The set $\{(r, \theta) : 1 \leq r \leq 2, \pi \leq \theta \leq 3\pi/2\}$ is a wedge of an annulus with inner radius 1 and outer radius 2, in the third (lower-left) quadrant. The Cartesian coordinates here are $\{(x, y) : x \leq 0, y \leq 0, 1 \leq x^2 + y^2 \leq 4\}$.

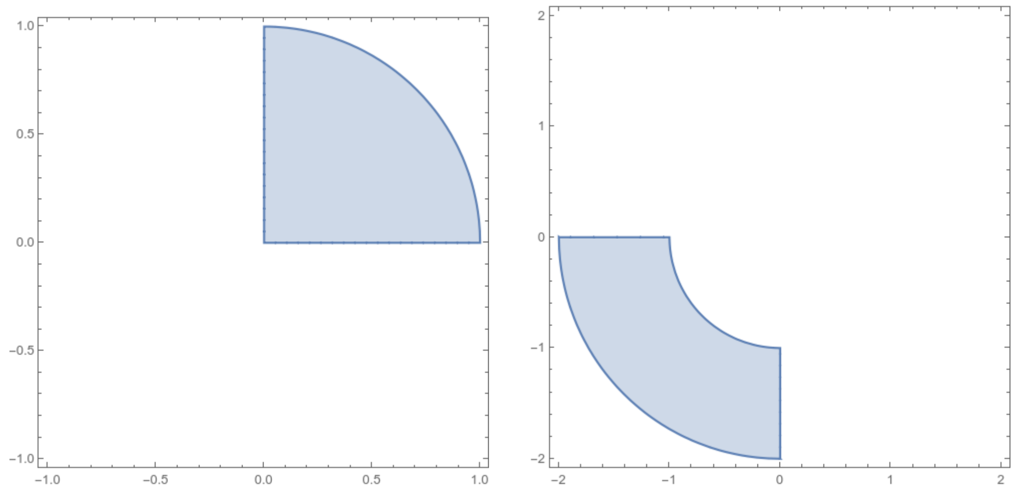


Figure 6.9: Left: The region $0 \leq r \leq 1, 0 \leq \theta \leq \pi/2$. Right: the region $1 \leq r \leq 2, \pi \leq \theta \leq 3\pi/2$.

Lines, on the other hand, tend to be obnoxious in polar. Through the origin they're not too bad:

Example 6.19. The polar equation for the line $y = 2x$ is $r \sin\theta = 2r\cos\theta$, which reduces to $\sin\theta = 2\cos\theta$. With a little more work, you can compute that this is equvalent to the line $\theta = \arctan(2)$.

But lines that don't go through the origin tend to produce genuinely obnoxious equations.

Example 6.20. Consider the line $y = 2x + 1$. Substituting in according to the rules in proposition 6.16 gives us $r \sin\theta = 2r\cos\theta + 1$. solving for r gives $r = \frac{1}{\sin(\theta) - 2\cos(\theta)}$.

And here are some much less obvious examples.

Example 6.21. $r = 2\cos\theta$ gives a circle centered at $(1, 0)$. In general, $r = 2n\cos(\theta)$ gives a circle centered at (n, θ). We can actually work this out from the substitution rules in proposition 6.16:

$$\begin{aligned} r &= 2\cos(\theta) \\ \sqrt{x^2+y^2} &= 2\cos(\arctan(y/x)) \\ &= 2\frac{x}{\sqrt{x^2+y^2}} \\ x^2+y^2 &= 2x \\ (x-1)^2+y^2 &= 1. \end{aligned}$$

Example 6.22. $r = 1+\cos(\theta)$ gives a shape called a *cardioid*. $r = \cos(2\theta)$ gives a flower. You can kind of work out why this would happen, but mostly they're just kinda pretty.

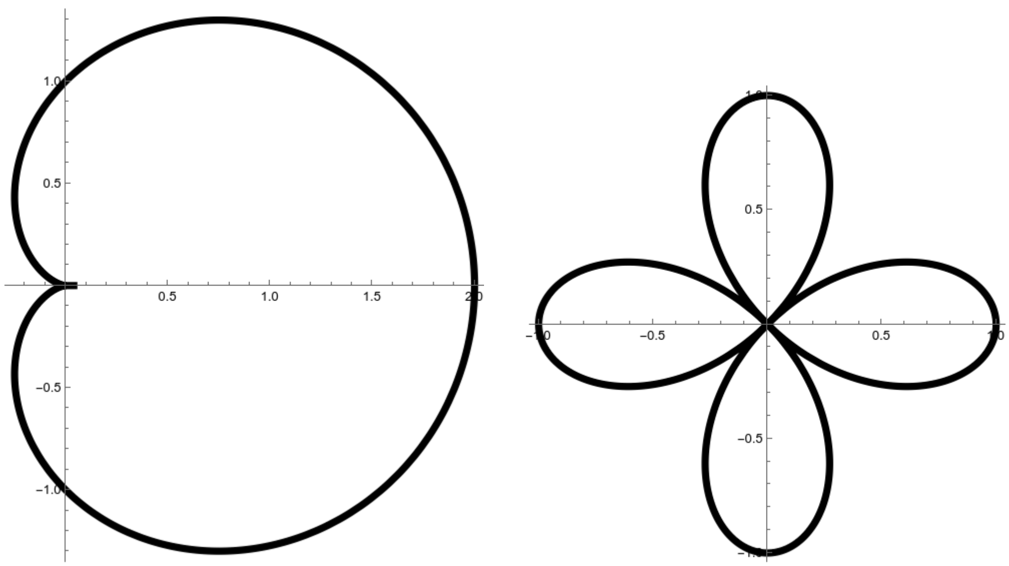


Figure 6.10: Left: $r = 1+\cos\theta$. Right: $r = \cos(2\theta)$.

6.2.1 Derivatives in Polar Coordinates

We can take the derivative of a polar function, but there's nothing really new there. The derivative measures the rate at which the radius r changes as the angle θ changes.

Example 6.23. Suppose we have $r = 2\cos\theta$. Then $\frac{dr}{d\theta} = -2\sin\theta$.

More interesting is looking for the equations of tangent lines. To do this we want to fall back on our theory of parametrization from section 6.1.1 to find a slope. We have formulas for x and y as (multivariable) functions of r and θ; but since here r is a function of θ, we can write everything in terms of θ.

Example 6.24. Suppose we want to find an equation for the tangent line to $r = 2\cos\theta$ at the point $r = \sqrt{3}, \theta = \pi/6$, which translates to $x = 3/2, y = \sqrt{3}/2$. We know that

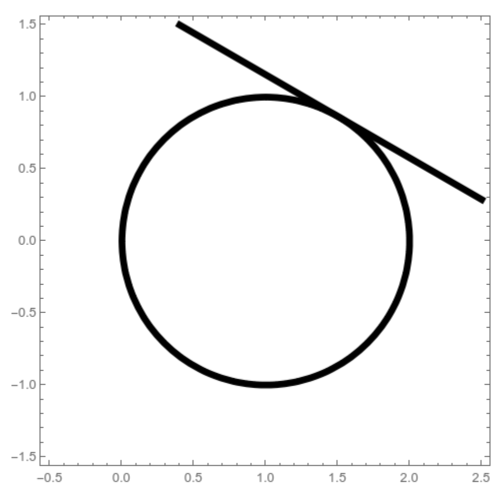
$$\begin{aligned}
x &= r\cos\theta = 2\cos^2\theta \\
y &= r\sin\theta = 2\cos\theta\sin\theta \\
\frac{dy}{dx} &= \frac{{}^{dy}/_{d\theta}}{{}^{dx}/_{d\theta}} = \frac{-2\sin^2(\theta) + 2\cos^2(\theta)}{-4\cos\theta\sin\theta} \\
&= \frac{\sin^2(\theta) - \cos^2(\theta)}{2\cos(\theta)\sin(\theta)} \\
\frac{dy}{dx}(\pi/6) &= \frac{1/4 - 3/4}{2 \cdot {}^{\sqrt{3}}/_2 \cdot {}^1/_2} = \frac{-1}{\sqrt{3}}
\end{aligned}$$

so the slope of our tangent line is $\frac{1}{\sqrt{3}}$ and the (Cartesian) equation is

$$y - \sqrt{3}/2 = \frac{-1}{\sqrt{3}}(x - 3/2).$$

If we really want to we can turn this into a polar equation:

$$r\sin\theta - \sqrt{3}/2 = \frac{-1}{\sqrt{3}}(r\cos(\theta) - 3/2).$$



6.2.2 Polar Integrals: areas and lengths

We've seen that polar coordinates tend to make circular equations become much simpler than their cartesian equivalents, but lines (and anything else rigid and rectangular) become much more complex.

We want to exploit this complexity reduction to make integrals of functions over circular regions easier. When we integrated over a rectangular region, we did this by dividing the

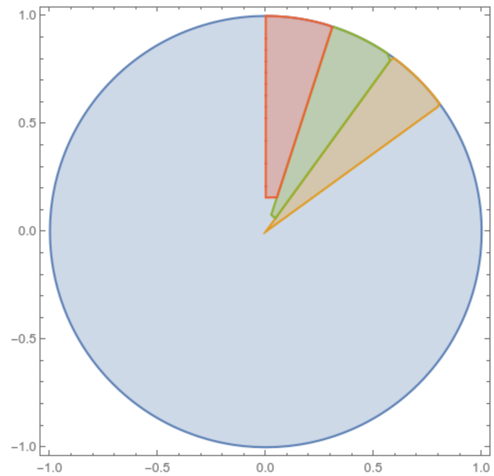


Figure 6.11: The subwedges for $2\pi/10 \leq \theta \leq 3\pi/10$, $3\pi/10 \leq \theta \leq 4\pi/10$, and $4\pi/10 \leq \theta \leq 5\pi/10$

region into rectangles. Using polar coordinates to integrate over a circular or wedge-like region, we'll divide the region into *subwedges*, as seen in in figure 6.11

What is the area of a wedge? If the wedge has outer radius r and is spanned by an angle $\Delta\theta$, then it is $\frac{\Delta\theta}{2\pi}$ of a circle of radius r. The area of such a circle is πr^2, so the area of our wedge is

$$\frac{\Delta\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \Delta\theta.$$

Thus if we have $r = f(\theta)$ on a shape going from $\theta = a$ to $\theta = b$, we can subdivide our region into n rectangles, and then our area is approximately

$$A \approx \sum_{i=1}^{n} \frac{1}{2} f(\theta_i)^2 \Delta\theta.$$

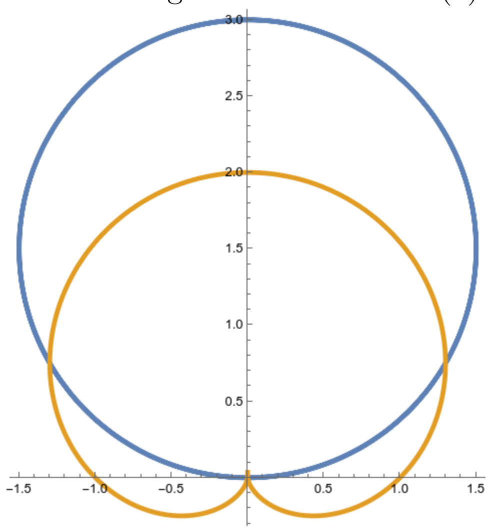
You should recognize this as a Riemann sum! Now that we have the explicit sum we no longer need to worry too much about where it came from; we know that Riemann sums become integrals as n approaches ∞, so we get

$$A = \int_a^b \frac{1}{2} f(\theta)^2 \, d\theta \qquad \text{or} \qquad A = \int_a^b \frac{1}{2} r^2 \, d\theta.$$

Example 6.25. Let's find the area inside one petal of the flower $r = \cos(2\theta)$. This is the range as θ goes from $-\pi/4$ to $\pi/4$. So we have the integral

$$\begin{aligned}
A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2}\cos^2(2\theta)\,d\theta \\
&= \int_{-\pi/4}^{\pi/4} \frac{1}{4}(1+\cos(4\theta)\,d\theta \\
&= \frac{\theta}{4} + \frac{\sin(4\theta)}{16}\Big|_{-\pi/4}^{\pi/4} \\
&= \frac{\pi}{16} + 0 - \frac{-\pi}{16} - 0 = \frac{\pi}{8}.
\end{aligned}$$

Example 6.26. Find the area of the region inside $r = 3\sin(\theta)$ and outside $r = 1+\sin\theta$.



These two curves intersect when

$$\begin{aligned}
3\sin(\theta) &= 1+\sin\theta \\
2\sin\theta &= 1 \\
\sin\theta &= 1/2 \\
\theta &= \pi/6 \text{ or } 5\pi/6.
\end{aligned}$$

We can find the area inside the blue curve $r = 3\sin\theta$ and then subtract off the area inside the yellow curve, so we get

$$\begin{aligned}
A &= \int_{\pi/6}^{5\pi/6} \frac{1}{2}(3\sin\theta)^2\,d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(1+\sin\theta)^2\,d\theta \\
&= \left(\frac{9\sqrt{3}}{8} + \frac{12\pi}{8}\right) - \left(\frac{9\sqrt{3}}{8} + \frac{4\pi}{8}\right) \\
&= \frac{12\pi}{8} - \frac{4\pi}{8} = \pi.
\end{aligned}$$

Notice something subtle is going on here: the first inegral doesn't compute the entire area of the blue circle, which would just be π. That's because we're only counting the area between $\theta = \pi/6$ and $\theta = 5\pi/6$; we're ignoring the thing slices at the bottom of the circle on either side.

Arc Length We can also compute the lengths of arcs in polar coordinates.

$$\begin{aligned}
L &= \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\
&= \int_a^b \sqrt{\left(\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)\right)^2 + \left(\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)\right)^2}\, d\theta \\
&= \int_a^b \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta.
\end{aligned}$$

Example 6.27. Find length of cardioid $1 + \sin\theta$.

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{r^2 + (dr/d\theta)^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{(1 + \sin\theta)^2 + \cos^2\theta}\, d\theta \\
&= \int_0^2 \pi\sqrt{2 + 2\sin\theta}\, d\theta \\
&= 8.
\end{aligned}$$