# Math 1232 Midterm Solutions

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- You will have 75 minutes for this test.
- You are not allowed to consult books or notes during the test, but you may use a one-page, one-sided, handwritten cheat sheet you have made for yourself ahead of time.
- You may not use a calculator.
- This test has eight questions, over five pages. **You should not answer all eight questions.**
  - The first two problems are three pages, representing topics M1 and M2. You should do both of them, and they are worth 30 points each.
  - The remaining six problems represent topics S1 through S6. You will be graded on your best three, with a few possible bonus points if you also do well on the others.
  - Doing three secondary topics well is much better than doing five or six poorly.
  - If you perform well on a question on this test it will update your mastery scores. Achieving a 27/30 on a major topic or 9/10 on a secondary topic will count as getting a 2 on a mastery quiz.

## Name:

## Recitation Section:

| | a | b | c |
|---|---|---|---|
| M1 | | | |
| M2 | | | |
| S1 | | S2 | |
| S3 | | S4 | |
| S5 | | S6 | |
| $\sum$ | | | /90 |

**Problem 1** (M1)**.** Compute the following using methods we have learned in class. Show enough work to justify your answers.

(a) Find the derivative of $y = (x^3 - x)^{\arctan(x)}$.

**Solution:**

$$\begin{aligned}
\ln|y| &= \arctan(x)\ln|x^3 - x| \\
\frac{y'}{y} &= \frac{\ln|x^3 - x|}{1 + x^2} + \frac{(3x^2 - 1)\arctan(x)}{x^3 - x} \\
y' &= y\left(\frac{\ln|x^3 - x|}{1 + x^2} + \frac{(3x^2 - 1)\arctan(x)}{x^3 - x}\right) \\
&= (x^3 - x)^{\arctan(x)}\left(\frac{\ln|x^3 - x|}{1 + x^2} + \frac{(3x^2 - 1)\arctan(x)}{x^3 - x}\right).
\end{aligned}$$

(b) Compute $\displaystyle\int_0^{1/\sqrt[3]{2}} \frac{x^2}{\sqrt{1 - x^6}}\,dx$

**Solution:** The obvious substitution is $u = 1 - x^6$ but that doesn't really work. After experimenting we see $u = x^3$ gives $du = 3x^2\,dx$ and thus

$$\begin{aligned}
\int_0^{1/\sqrt[3]{2}} \frac{x^2}{\sqrt{1 - x^6}}\,dx &= \int_0^{1/2} \frac{1}{3}\frac{1}{\sqrt{1 - u^2}}\,du \\
&= \frac{1}{3}\arcsin(u)\Big|_0^{1/2} \\
&= \frac{1}{3}(\arcsin(1/2) - \arcsin(0)) = \frac{1}{3}(\pi/6 - 0) = \frac{\pi}{18}.
\end{aligned}$$

Alternatively we could compute that

$$\int \frac{x^2}{\sqrt{1 - x^6}}\,dx = \frac{1}{3}\arcsin(x^3)$$

and thus

$$\begin{aligned}
\int_0^{1/\sqrt[3]{2}} \frac{x^2}{\sqrt{1 - x^6}}\,dx &= \frac{1}{3}\arcsin(x^3)\Big|_0^{1/\sqrt[3]{2}} \\
&= \frac{1}{3}(\arcsin(1/2) - \arcsin(0)) = \frac{1}{3}(\pi/6 - 0) = \frac{\pi}{18}.
\end{aligned}$$

(c) Compute $\displaystyle\int \frac{3e^{3x} + e^x}{e^{3x} + e^x}\,dx$.

**Solution:** We take $u = e^{3x} + e^x$, so $\frac{du}{dx} = 3e^{3x} + e^x$ and we get $dx = \frac{du}{3e^{3x} + e^x}$. Then

$$\begin{aligned}
\int \frac{3e^{3x} + e^x}{e^{3x} + e^x}\,dx &= \int \frac{3e^{3x} + e^x}{u}\frac{du}{3e^{3x} + e^x} \\
&= \int \frac{du}{u} = \ln|u| + C \\
&= \ln(e^{3x} + e^x) + C.
\end{aligned}$$

**Problem 2** (M2)**.** Compute the following integrals using methods we have learned in class. Show enough work to justify your answers.

(a) $\displaystyle\int x^2 e^{3x}\,dx$

**Solution:** We use integration by parts. Take $u = x^2, dv = e^{3x}\,dx$ so $du = 2x\,dx, v = e^{3x}/3$. Then

$$\begin{aligned}\int x^2 e^{3x}\,dx &= x^2\frac{e^{3x}}{3} - \int \frac{2}{3}xe^{3x}\,dx \\ &= \frac{x^2e^{3x}}{3} - \left(\frac{2}{9}xe^{3x} - \int \frac{2}{9}e^{3x}\right) \\ &= \frac{x^2e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2e^{3x}}{27} + C.\end{aligned}$$

(b) $\displaystyle\int \frac{3x^2-7x+8}{x(x-2)^2}\,dx =$

**Solution:** We set up

$$\begin{aligned}\frac{3x^2-7x+8}{x(x-2)^2} &= \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \\ 3x^2-7x+8 &= A(x-2)^2 + B(x-2)x + Cx\end{aligned}$$

$$\begin{array}{lll} 0: & 8 = A(-2)^2 = 4A & \Rightarrow A = 2 \\ 2: & 6 = 2x & \Rightarrow C = 3 \\ 1: & 4 = 2 - B + 3 & \Rightarrow B = 1. \end{array}$$

So we have

$$\begin{aligned}\int \frac{3x^2-7x+8}{x(x-2)^2}\,dx &= \int \frac{2}{x} + \frac{1}{x-2} + \frac{3}{(x-2)^2}\,dx \\ &= 2\ln|x| + \ln|x-2| - \frac{3}{x-2} + C.\end{aligned}$$

(c) $\displaystyle\int \frac{x^3}{\sqrt{x^2+4}}\,dx =$

**Solution:** Here we should do a trigonometric substitution. We take $x = 2\tan(\theta)$, so we have $dx = 2\sec^2(\theta)\,d\theta$. Then

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+4}}\,dx &= \int \frac{8\tan^3(\theta)}{\sqrt{4\tan^2(\theta)+4}}2\sec^2(\theta)\,d\theta \\ &= \int \frac{8\tan^3(\theta)}{\sqrt{4\sec^2(\theta)}}2\sec^2(\theta)\,d\theta \\ &= \int \frac{8\tan^3(\theta)}{2\sec(\theta)}2\sec^2(\theta)\,d\theta \\ &= \int 8\sec(\theta)\tan^3(\theta)\,d\theta.\end{aligned}$$

Now we need to use some trigonometric identities to do this integral. We know that things work if we have a single tangent function left, so we set $u = \sec(\theta)$, $du = \sec(\theta)\tan(\theta)\,d\theta$, and get

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+4}}\,dx &= \int 8\sec(\theta)\tan^3(\theta)\,d\theta \\ &= \int 8\sec(\theta)\tan(\theta)(\sec^2(\theta)-1)\,d\theta \\ &= 8\int u^2 - 1\,du = \frac{8}{3}u^3 - 8u + C = \frac{8}{3}\sec^3(\theta) - 8\sec(\theta) + C.\end{aligned}$$

Now we just need to figure out what $\sec(\theta)$ is. We know $\tan(\theta) = x/2$, so we get a triangle with opposite side $x$, adjacent side 2, and hypotenuse $\sqrt{x^2+4}$. Then we see that $\sec(\theta) = \frac{\sqrt{x^2+4}}{2}$, and thus the integral is

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx = \frac{(x^2+4)^{3/2}}{3} - 4\sqrt{x^2+4} + C.$$

**Problem 3** (S1)**.** Let $f(x) = e^{(x^3+x)} + x - 1$. Find $(f^{-1})'(e^2)$.

**Solution:** Plugging in numbers, we see $f(0) = 0$ but $f(1) = e^2$, so $f^{-1}(e^2) = 1$. We compute

$$\begin{aligned} f'(x) &= e^{x^3+x}(3x^2+1) + 1 \\ f'(1) &= e^2 \cdot (3+1) + 1 = 4e^2 + 1 \end{aligned}$$

and thus by the Inverse Function Theorem,

$$\begin{aligned} (f^{-1})'(e^2) &= \frac{1}{f'(f^{-1}(e^2))} \\ &= \frac{1}{f'(1)} = \frac{1}{4e^2+1} \approx 0.033. \end{aligned}$$

**Problem 4** (S2)**.** Find $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{\ln(x) - x}$.

**Solution:** So I actually typo'd this one. I originally intended to ask this:

$$\begin{aligned} \lim_{x\to 0} \frac{e^x - 1 - x^{\nearrow 0}}{\ln(1+x) - x_{\searrow 0}} &=^{\text{L'H}} \lim_{x\to 0} \frac{e^x - 1^{\nearrow 0}}{1/(1+x) - 1_{\searrow 0}} \\ &=^{\text{L'H}} \lim_{x\to 0} \frac{e^{x\nearrow 1}}{-1/(1+x)^2{}_{\searrow -1}} = -1. \end{aligned}$$

But in fact with the question I wrote, the correct answer is something like

$$\lim_{x\to 0} \frac{e^x - 1 - x^{\nearrow 0}}{\ln(x) - x_{\searrow -\infty}} = 0.$$

**Problem 5** (S3)**.** Use the Trapezoid rule with six intervals to estimate $\int_{-4}^{2} x^2 + 1\,dx$.

**Solution:**

$$\begin{aligned} \int_{-4}^{2} x^2 + 1\,dx &\approx \frac{f(-4)+f(-3)}{2} + \frac{f(-3)+f(-2)}{2} + \frac{f(-2)+f(-1)}{2} \\ &\quad + \frac{f(-1)+f(0)}{2} + \frac{f(0)+f(1)}{2} + \frac{f(1)+f(2)}{2} \\ &= \frac{17+10}{2} + \frac{10+5}{2} + \frac{5+2}{2} + \frac{2+1}{2} + \frac{1+2}{2} + \frac{2+5}{2} \\ &= \frac{1}{2}(27+15+7+3+3+7) = \frac{62}{2} = 31. \end{aligned}$$

Alternatively, we could write

$$\begin{aligned} \int_{-4}^{2} x^2 + 1\,dx &\approx \frac{1}{2}\Big(f(-4) + 2f(-3) + 2f(-2) + 2f(-1) + 2f(0) + 2f(1) + f(2)\Big) \\ &= \frac{1}{2}\Big(17 + 20 + 10 + 4 + 2 + 4 + 5\Big) = \frac{1}{2}\cdot 62 = 31. \end{aligned}$$

**Problem 6** (S4)**.** Compute $\int_0^{+\infty} \frac{1}{x^2}\,dx$.

**Solution:** This is improper in two ways: there's a singularity at 0, and it goes to $+\infty$. Thus we have to compute

$$\begin{aligned}\int_0^{+\infty} \frac{1}{x^2}\,dx = \int_0^1 \frac{1}{x^2}\,dx + \int_1^{+\infty} \frac{1}{x^2}\,dx\\
&= \lim_{s\to 0^+} \int_s^1 \frac{1}{x^2}\,dx + \lim_{t\to+\infty} \int_1^t \frac{1}{x^2}\,dx\\
&= \lim_{s\to 0^+} \frac{-1}{x}\bigg|_s^1 + \lim_{t\to+\infty} \frac{-1}{x}\bigg|_1^t\\
&= \lim_{x\to 0^+} \left(-1+\frac{1}{s}\right) + \lim_{t\to+\infty}\left(\frac{-1}{t}+1\right).\end{aligned}$$

The second limit is 1, but the first limit is $+\infty$, so the whole limit is $+\infty$ and thus the integral does not converge.

**Problem 7** (S5)**.** Compute the arc length of the curve $y = \frac{1}{27}(9x^2+6)^{3/2}$ as $x$ varies from 2 to 4.

**Solution:** We have $y' = x\sqrt{9x^2+6}$, and thus

$$\begin{aligned}L &= \int_2^4 \sqrt{1+x^2(9x^2+6)}\,dx = \int_2^4 \sqrt{1+6x^2+9x^4}\,dx\\
&= \int_2^4 3x^2+1 = x^3+x\big|_2^4 = 64+4-8-2 = 58\end{aligned}$$

**Problem 8** (S6)**.** Find the function that is the (specific) solution to the initial value problem $y' = (y^2+1)(x^2+1)$ if $y(0)=1$.

**Solution:**

$$\begin{aligned}\frac{dy}{y^2+1} &= (x^2+1)\,dx\\
\int \frac{dy}{y^2+1} &= \int (x^2+1)\,dx\\
\arctan(y) &= \frac{x^3}{3}+x+C\\
y &= \tan\left(\frac{x^3}{3}+x+C\right).\end{aligned}$$

So find a specific solution, we plug in and get

$$\begin{aligned}1 &= \tan(0+0+C)\\
C &= \arctan(1) = \pi/4\end{aligned}$$

and thus

$$y = \tan\left(\frac{x^3}{3}+x+\pi/4\right).$$