Math 1231 Practice Midterm Solutions

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- These are the instructions you will see on the real test, next week. I include them here so you know what to expect.
- You will have 75 minutes for this test.
- You are not allowed to consult books or notes during the test, but you may use a one-page, one-sided, handwritten cheat sheet you have made for yourself ahead of time.
- You may not use a calculator.
- This test has eight questions, over five pages. **You should not answer all seven questions.**
  - The first two problems are two pages, representing topics M1 and M2. You should do both of them, and they are worth 30 points each.
  - The remaining six problems represent topics S1 through S6. You will be graded on your best three, with a few possible bonus points if you also do well on the other two.
  - Doing three secondary topics well is much better than doing five poorly.
  - If you perform well on a question on this test it will update your mastery scores. Achieving a 27/30 on a major topic or 9/10 on a secondary topic will count as getting a 2 on a mastery quiz.

Problem 1 (M1)**.** Compute the following using methods we have learned in class. Show enough work to justify your answers.

(a) Find the tangent line to $h(x) = \arcsin(e^x)$ at $\ln(1/2)$.

Solution: We have $h'(x) = \frac{1}{\sqrt{1-e^{2x}}} \cdot e^x$, so $h'(\ln(1/2)) = \frac{e^{\ln 1/2}}{\sqrt{1-e^{2\ln(1/2)}}} = \frac{1/2}{\sqrt{1-1/4}} = \frac{1}{\sqrt{3}}$. We also have $h(\ln(1/2)) = \arcsin(1/2) = \pi/6$.

Thus the equation of the tangent line is

$$y - \pi/6 = \frac{1}{\sqrt{3}}(x - \ln(1/2)).$$

(b) $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\, dx =$

Solution: We take $u = 1/x$ so $du = \frac{-1}{x^2}\, dx$. Then

$$\begin{aligned}\int_1^2 \frac{e^{1/x}}{x^2}\, dx &= \int_1^{1/2} -e^u\, du \\ &= -e^u\big|_1^{1/2} = -e^{1/2} + e^1 = e - \sqrt{e}.\end{aligned}$$

(c) $\displaystyle\int \frac{\cos(x)\sin(x)}{1+\cos^4(x)}\, dx =$

Solution: We can take $u = \cos(x)$ so that $du = -\sin(x)\, dx$. Then

$$\int \frac{\cos(x)\sin(x)}{1+\cos^4(x)}\, dx = \int \frac{-u}{1+u^4}\, du$$

Then we can set $v = u^2$ so that $dv = 2u\, du$ and we get

$$\begin{aligned}\int \frac{-u}{1+u^4}\, du &= \int \frac{-1}{2}\frac{1}{1+v^2}\, dv = \frac{-1}{2}\arctan(v) + C \\ &= \frac{-1}{2}\arctan(u^2) + C = \frac{-1}{2}\arctan(\cos^2(x)) + C.\end{aligned}$$

Problem 2 (M2)**.** Compute the following integrals using methods we have learned in class. Show enough work to justify your answers.

(a) $\displaystyle\int \frac{2x+1}{\sqrt{x^2-1}}\, dx$

Solution: Since we see $\sqrt{x^2-1}$ we want to try a trig substitution. (You might try $u = x^2 - 1$ first, which almost works, but doesn't quite). So we set $x = \sec\theta$ and $dx = \sec\theta\tan\theta\, d\theta$. We have

$$\begin{aligned}\int \frac{2x+1}{\sqrt{x^2-1}}\, dx &= \int \frac{2\sec\theta + 1}{\sqrt{\sec^2\theta - 1}} \sec\theta\tan\theta\, d\theta \\ &= \int \frac{2\sec^2\theta\tan\theta + \sec\theta\tan\theta}{\tan\theta}\, d\theta \\ &= \int 2\sec^2\theta + \sec\theta\, d\theta \\ &= 2\tan\theta + \ln|\sec\theta + \tan\theta| + C.\end{aligned}$$

If $\sec\theta = x$ then θ is in a triangle with hypotenuse x and adjacent side 1 and thus opposite side $\sqrt{x^2-1}$. Thus $\tan\theta = \sqrt{x^2-1}$. This is good, since this formula appeared in our original question, and we see that

$$\int \frac{2x+1}{\sqrt{x^2-1}}\,dx = 2\sqrt{x^2-1} + \ln\left|x+\sqrt{x^2-1}\right| + C.$$

(b) $\displaystyle\int x\sec^2 x\,dx$

Solution: We use integration by parts. Take $u = x, dv = \sec^2 x\,dx$ so $du = dx, v = \tan x$. Then

$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x + \ln|\cos x| + C.$$

(c) $\displaystyle\int_0^1 \frac{3x^2-6x+1}{(x^2-x-1)(x-2)}\,dx$

Solution: We use a partial fractions decomposition.

$$\begin{aligned}\frac{3x^2-6x+1}{(x^2-x-1)(x-2)} &= \frac{A}{x-2} + \frac{Bx+C}{x^2-x-1}\\ 3x^2-6x+1 &= A(x^2-x-1) + (Bx+C)(x-2).\end{aligned}$$

Plugging in $x=2$ gives us that $1 = A$. Plugging in $x = 0$ gives $1 = -A-2C = -1-2C$ and thus $C = -1$. Then plugging in $x = 1$ gives $-2 = -A-B-C = -1-B+1$ and thus $B = 2$. So we have

$$\begin{aligned}\int_0^1 \frac{3x^2-6x+1}{(x^2-x-1)(x-2)}\,dx &= \int_0^1 \frac{1}{x-2} + \frac{2x-1}{x^2-x-1}\,dx\\ &= \left(\ln|x-2| + \ln|x^2-x-1|\right)\Big|_0^1\\ &= \ln(1) + \ln(1) - \ln(2) - \ln(1) = -\ln(2).\end{aligned}$$

Problem 3 (S1)**.** Let $f(x) = \sqrt[3]{x^5+x^4+x^3+x^2+2x}$. Find $(f^{-1})'(4)$.

Solution: Plugging in numbers, we see that $f(2) = \sqrt[3]{32+16+8+4+4} = \sqrt[3]{64} = 4$. Then by the Inverse Function Theorem we have $(f^{-1})'(4) = \frac{1}{f'(2)}$. But

$$\begin{aligned}f'(x) &= \frac{1}{3}\left(x^5+x^4+x^3+x^2+2x\right)^{-2/3}\left(5x^4+4x^3+3x^2+2x+2\right)\\ f'(2) &= \frac{1}{3}(64)^{-2/3}\left(80+32+12+4+2\right) = \frac{130}{48} = \frac{65}{24}.\end{aligned}$$

Thus by the inverse function theorem we have

$$(f^{-1})'(4) = \frac{24}{65}.$$

Problem 4 (S2)**.** Find $\lim_{x\to 0}\frac{2\sin(x)-\sin(2x)}{x-\sin(x)}$.

Solution: $\lim_{x\to 0} 2\sin(x) - \sin(2x) = 0 - 0 = 0$, and $\lim_{x\to 0} x - \sin(x) = 0$, so we can use L'Hospital's Rule.

$$\begin{aligned}
\lim_{x\to 0} \frac{2\sin(x) - \sin(2x)}{x - \sin(x)} &=^{\text{L'H}} \lim_{x\to 0} \frac{2\cos(x) - 2\cos(2x)^{\nearrow 0}}{1 - \cos(x)_{\searrow 0}} \\
&=^{\text{L'H}} \lim_{x\to 0} \frac{-2\sin(x) + 4\sin(2x)^{\nearrow 0}}{\sin(x)_{\searrow 0}} \\
&=^{\text{L'H}} \lim_{x\to 0} \frac{-2\cos(x) + 8\cos(2x)}{\cos(x)} = \frac{6}{1} = 6.
\end{aligned}$$

Problem 5 (S3)**.** Use Simpson's rule and six intervals to estimate $\int_0^6 x^4\,dx$. Give an upper bound for the error on this approximation.

Solution:

$$\begin{aligned}
\int_0^6 x^4\,dx &\approx \frac{1}{3}\left(0^4 + 4\cdot 1^4 + 2\cdot 2^4 + 4\cdot 3^4 + 2\cdot 4^4 + 4\cdot 5^4 + 6^4\right) \\
&= \frac{1}{3}\left(0 + 4 + 32 + 324 + 512 + 2500 + 1296\right) = \frac{4668}{3} = 1556.
\end{aligned}$$

To find the error: if $f(x) = x^4$ then $f''''(x) = 24$, so we can take $L = 24$. Then we have the formula

$$|E_S| \leq \frac{L(b-a)^5}{180n^4} = \frac{24\cdot 6^5}{180\cdot 6^4} = \frac{24\cdot 6}{180} = \frac{4}{5}.$$

So the error in this approximation is less than or equal to $4/5$. If we work things out exactly, we see $\int_0^6 x^4\,dx = 1555.2$, so the error is in fact $4/5$ exactly.

Problem 6 (S4)**.** Compute $\displaystyle\int_1^{10} \frac{1}{\sqrt[3]{x-2}}\,dx$.

Solution: We must split the integral up into two parts:

$$\begin{aligned}
\int_1^{10} \frac{1}{\sqrt[3]{x-2}}\,dx &= \int_1^2 \frac{1}{\sqrt[3]{x-2}}\,dx + \int_2^{10} \frac{1}{\sqrt[3]{x-2}}\,dx \\
&= \lim_{s\to 2^-} \int_1^s \frac{dx}{\sqrt[3]{x-2}} + \lim_{t\to 2^+} \int_t^{10} \frac{dx}{\sqrt[3]{x-2}} \\
&= \lim_{s\to 2^-} \frac{3}{2}(x-2)^{2/3}\Big|_1^s + \lim_{t\to 2^+} \frac{3}{2}(x-2)^{2/3}\Big|_t^{10} \\
&= \left(\lim_{s\to 2^-} \frac{3(s-2)^{2/3}}{2} - \frac{3}{2}\right) + \left(\lim_{t\to 2^+} \frac{3\cdot 8^{2/3}}{2} - \frac{3(t-2)^{2/3}}{2}\right) \\
&= \frac{3}{2}\cdot 0 - \frac{3}{2} + \frac{12}{2} - \frac{3}{2}\cdot 0 = \frac{9}{2}.
\end{aligned}$$

Problem 7 (S5)**.** Find the surface area of the surface obtained by rotating $y = \sqrt{5+4x}$ for $-1 \leq x \leq 1$ about the x-axis.

Solution: We have $y' = \frac{1}{2}(5+4x)^{-1/2} \cdot 4 = \frac{2}{\sqrt{5+4x}}$, so $ds = \sqrt{1 + \frac{4}{5+4x}} dx$. Then

$$\begin{aligned}
A &= \int_{-1}^{1} 2\pi y \, ds = 2\pi \int_{-1}^{1} \sqrt{5+4x}\sqrt{1 + \frac{4}{5+4x}} dx \\
&= 2\pi \int_{-1}^{1} \sqrt{5+4x+4} \, dx = 2\pi \int_{-1}^{1} \sqrt{9+4x} \, dx \\
&= 2\pi \left(\frac{2}{3}(9+4x)^{3/2} \cdot \frac{1}{4} \right) \Big|_{-1}^{1} = 2\pi \left(\frac{1}{6} 13\sqrt{13} - \frac{1}{6} 5\sqrt{5} \right) = \frac{\pi}{3} \left(13\sqrt{13} - 5\sqrt{5} \right).
\end{aligned}$$

Problem 8 (S6)**.** Find a (specific) solution to the initial value problem $y'/x - y = 1$ if $y(0) = 3$

Solution:

$$\begin{aligned}
y'/x &= 1 + y \\
\frac{dy}{1+y} &= x \, dx \\
\ln|1+y| x^2/2 + C \\
1 + y &= e^{x^2/2} e^C \\
y &= Ke^{x^2/2} - 1 \\
3 &= K - 1 \Rightarrow K = 4 \\
y &= 4e^{x^2/2} - 1.
\end{aligned}$$