1 Transcendental Functions

1.1 Invertible functions

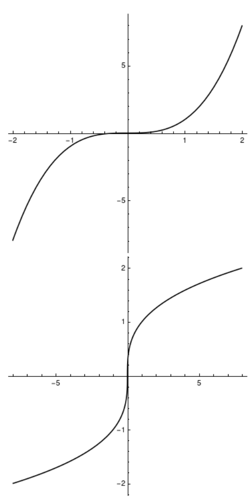
Recall that a *function* is a rule that takes an input and assigns a specific output. Sometimes we want to undo this process. This is in fact a natural question; "What do I have to do if I want to get X" is a pretty common thought process. So our goal is: given a function f, given $f(x)$, can we find x?

Definition 1.1. If f is a function and $(g \circ f)(x) = x$ for every x in the domain of f, then we say g is an *inverse* of f.

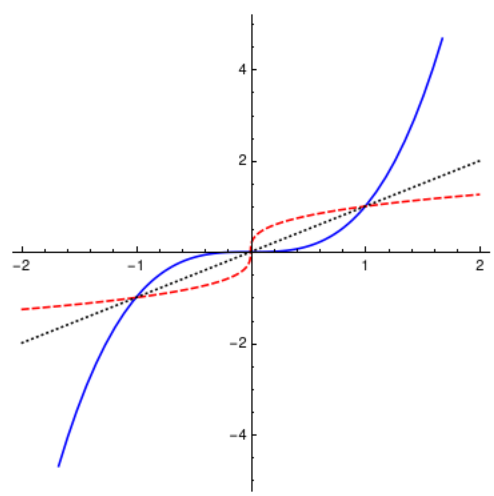
Example 1.2.

- If $f(x) = x$ then $g(y) = y$ is an inverse to f.
- If $f(x) = 5x + 3$ then $g(y) = (y - 3)/5$ is an inverse to f.
- If $f(x) = x^3$ then $g(y) = \sqrt[3]{y}$ is an inverse to f.

Graphically, the graph of f^{-1} looks like the graph of f flipped across the line $y = x$, which makes sense, since a point (x, y) on the graph of f should correspond to a point (y, x) on the graph of f^{-1}.



Top: x^3. Bottom: $\sqrt[3]{x}$. See how they mirror each ohter.



The graph of x^3 in solid blue, and the graph of $\sqrt[3]{x}$ in dashed red. Notice they are mirrored across the dotted black line $y = x$.

Remark 1.3. A given function f has at most one inverse—if f has an inverse at all, then that means "for any y, find the x where $f(x) = y$" is a well-defined rule.

If g is an inverse to f, then the domain of g is the image of f and the domain of f is the image of g.

Computing $f^{-1}(y)$ is the same as solving the equation $f(x) = y$.

Unfortunately, we can't always find these inverses. For instance, if you know that $x^2 = 9$, you don't know for sure what x is: it could be 3 or -3. Similarly, if you know $\sin(x) = 0$, then x could be $n\pi$ for any integer n. The fundamental problem here is that there are some outputs that are generated by more than one input.

Definition 1.4. A function f is *1-1* or *one-to-one* (or *injective*) if, whenever $f(a) = f(b)$, we know that $a = b$.

Example 1.5. Functions which are 1-1:

- $f(x) = x$. If $f(a) = f(b)$ then $a = b$ by definition.
- $f(x) = x^3$. If $f(a) = f(b)$ then $a^3 = b^3$, and then $(a/b)^3 = 1$ so $a/b = 1$ and $a = b$.
- $f(x) = \sqrt{x}$. If $f(a) = f(b)$ then $\sqrt{a} = \sqrt{b}$ so $|a| = |b|$. But $a, b \geq 0$ since they're in the domain of f, and thus $a = b$.

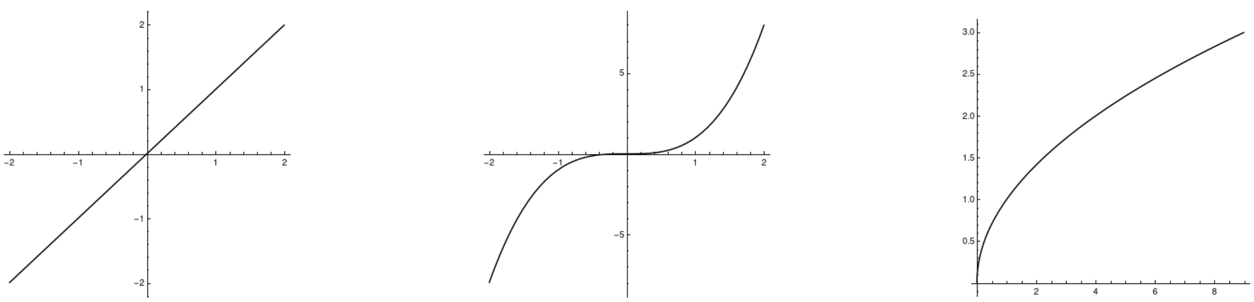


Figure 1.1: Some one-to-one functions: $f(x) = x, f(x) = x^3, f(x) = \sqrt{x}$

Functions which are not 1-1:

- $f(x) = x^2$, since $f(-1) = f(1)$.
- $f(x) = |x|$, since $f(-2) = f(2)$.
- $\sin(x)$, since $\sin(0) = \sin(\pi)$.
- $f(x) = 3$, since $f(a) = f(b) = 3$ for any real numbers a and b.

We might also want to think about what being one-to-one means for the graph of a function. We can't have two inputs with the same output, which means we can't have the same horizontal position at two different points.

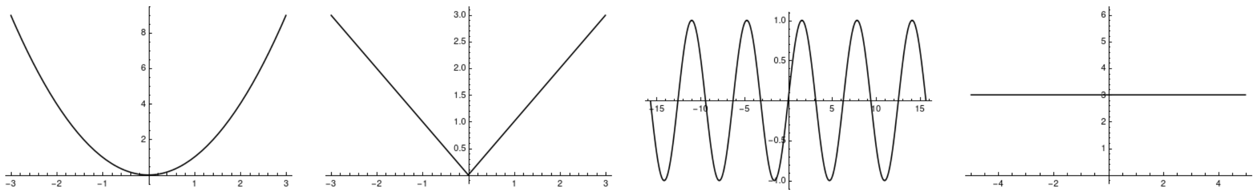


Figure 1.2: Some not one-to-one functions: $f(x) = x^2, f(x) = |x|, f(x) = \sin(x), f(x) = 3$.

Proposition 1.6 (Horizontal Line Test)**.** *A function f is 1-1 if and only if any horizontal line will intersect its graph in at most one point.*

We can see this on the graphs above: all the one-to-one graphs pass the horizontal line test, and all the not one-to-one graphs fail it. We can also interpret it in terms of the reflection property: a function passing the horizontal line test is the same as its reflection/inverse passing the vertical line test.

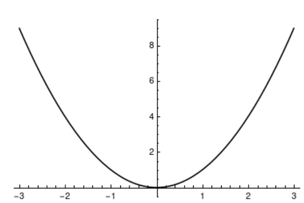
We already saw that every function with an inverse must be one-to-one, since otherwise there's not a unique answer to the inverse question. Less obvious is that being 1-1 is enough to be invertible, but it's true.

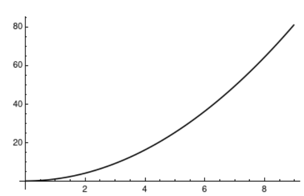
Proposition 1.7. *If f is a 1-1 function with domain A and image B, then there is a function f^{-1} with domain B and image A which is an inverse to f.*

Thus we know now exactly which functions have inverses. However, a lot of functions we would *like* to invert are not one-to-one, which causes a problem. We can often solve this problem by restricting the domain of a function to force it to become one-to-one.

Example 1.8.

We want $\sqrt{x}$ to be the inverse of x^2, but it really isn't. We know that $\sqrt{x^2} = x$ *if* $x \geq 0$, but if x is a negative number this doesn't work. The function $f(x) = x^2$ isn't one-to-one, and thus isn't invertible.

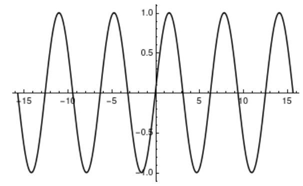




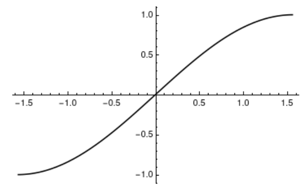
But consider the function $f(x) = x^2$ on the domain $[0, +\infty)$. We can prove this function is one-to-one: if $f(a) = f(b)$ then $a^2 = b^2$ so $a = \pm b$. But both $a, b \geq 0$ so $a = b$. And in fact $\sqrt{x}$ is an inverse to the function $f(x) = x^2$ defined on the domain $[0, +\infty)$.

Example 1.9.

We saw that $\sin(x)$ isn't invertible. For instance, $\sin(n\pi) = 0$ for any whole number n.



But if we consider the function $\sin(x)$ restricted to the domain $[-\pi/2, \pi/2]$, it is in fact one-to-one. If we look at the unit circle, we see that as x varies from $-\pi/2$ to $\pi/2$, the y coordinate on the unit circle is always increasing, and so never repeats itself.



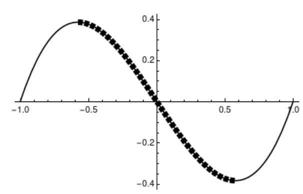
Thus we can find an inverse to the sine function on the domain $[-\pi/2, \pi/2]$; we will discuss this further in section 1.5.

We can find the inverse to a function by writing the equation $y = f(x)$ and solving for x as a function of y. (Sometimes we instead write $x = f(y)$ and solve for y as a function of x; it depends on how we're thinking of the function and what we plan to use it for.) This is also a good way to prove that f is one-to-one.

Example 1.10. Let $f(x) = x^4$ with domain $(-\infty, 0]$. Then we have $y = x^4 \Rightarrow x = \pm\sqrt[4]{y}$. But we know that $x < 0$ so $x = -\sqrt[4]{y}$, and thus $g(y) = -\sqrt[4]{y}$ is an inverse for f.

Example 1.11.

Take $f(x) = x^3 - x$. This function is clearly not one-to-one, since $f(1) = f(0) = f(-1) = 0$. But we can split it up into intervals where it is one-to-one. Looking at the graph, it seems natural to split it up at the critical points. And this suggests we should use calculus to study our inverse function problem.



1.1.1 Calculus of inverse functions

Now that we understand inverse functions as functions, we'd like to see what calculus can tell us about them.

Proposition 1.12. *If f is one-to-one and continuous at a, then f^{-1} is continuous at $f(a)$.*

If f is one-to-one and continuous, then f^{-1} is continuous.

We'd really like to know about the derivatives of inverse functions. We can work out what they are with some quick sketched arguments, and then can prove the answer rigorously once we know what we're looking for.

First, the argument by "it looks nice in the notation": we can rephrase this theorem as saying that

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Second, if we already know that both functions are differentiable, we can use implicit differentiation:

$$\begin{aligned} f^{-1}(f(x)) &= x \\ (f^{-1})'(f(x)) \cdot f'(x) &= 1 \\ (f^{-1})'(f(x)) &= \frac{1}{f'(x)}. \end{aligned}$$

Writing $x = f^{-1}(a)$, or equivalently $a = f(x)$, gives our statement.

Theorem 1.13 (Inverse Function Theorem)**.** *If f is a one-to-one differentiable function, and $f'(f^{-1}(a)) \neq 0$, then $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$.*

Proof. Set $y = f^{-1}(x)$ and $b = f^{-1}(a)$. Then

$$\begin{aligned} (f^{-1})'(a) &= \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a} \\ &= \lim_{y \to b} \frac{y - b}{f(y) - f(b)} \\ &= \lim_{y \to b} \frac{1}{\frac{f(y) - f(b)}{y - b}} \\ &= \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))}. \end{aligned}$$

□

Graphically, this result tells us that the tangent line to f^{-1} at a point has a slope reciprocal to the slope of the tangent line to f at that same point. Really, the tangent line is just being reflected with the graph of the function.

Example 1.14. Let $f(x) = x^n$ on $[0, +\infty)$; then $f^{-1}(x) = \sqrt[n]{x}$. Our formula gives

$$\begin{aligned} (f^{-1})'(a) &= \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(\sqrt[n]{(a)})} \\ &= \frac{1}{n(\sqrt[n]{(a)})^{n-1}} = \frac{1}{na^{(n-1)/n}} = \frac{1}{n} a^{(1-n)/n} = \frac{1}{n} a^{\frac{1}{n} - 1}. \end{aligned}$$

Though at first this didn't look like our original answer, it is the same as the formula we had before.

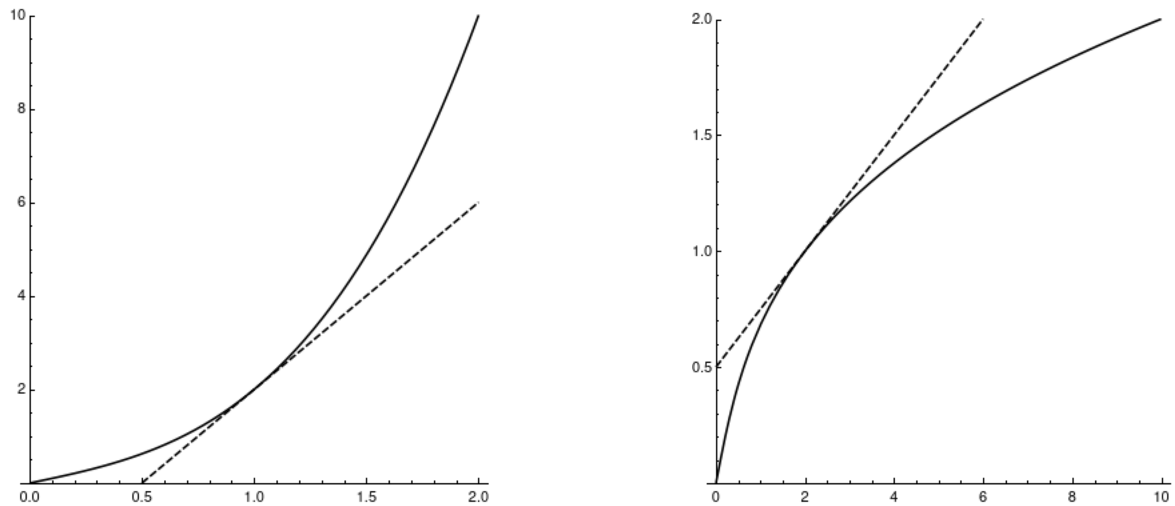


Figure 1.3: Left: the graph of $f(x) = x^3 + x$ with the tangent line at $(x, y) = (1, 2)$. Right: the graph of $f^{-1}(y)$ with the tangent line at $(y, x) = (2, 1)$.

Example 1.15. Let $f(x) = \sqrt[3]{5x^2 + 7}$. What is $(f^{-1})'(3)$?

Well, we have $(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))}$. We know that $f'(x) = \frac{1}{3}(5x^2 + 7)^{-2/3} \cdot 10x$, and we can work out that $f(2) = \sqrt[3]{20 + 7} = 3$ (by plugging in small integers until one works). Thus $f^{-1}(3) = 2$, and so we have

$$(f^{-1})'(3) = \frac{1}{\frac{1}{3}(27)^{-2/3} \cdot 20} = \frac{3 \cdot 9}{20} = \frac{27}{20}.$$

1.2 The exponential and the logarithm

In this section we'll look at a specific, extremely important example: the exponential function $e^x = \exp(x)$ and its inverse the logarithm.

1.2.1 The Exponential

By now we should be familiar with the function $f(x) = x^n$. It's simple to define x^n when n is a positive integer, as $x \cdot x \cdot \cdots \cdot x$. It's now clear that we defined $x^{1/n}$ as the inverse function to x^n, with domain restricted to positive numbers in the case n is even and thus x^n is not one-to-one. But can we make sense of x^r where r is any real number? What would it mean to write $2^{\sqrt{2}}$?

The answer would presumably be between 2 and 4. And also between $2^{1.4}$ and $2^{1.5}$. And between $2^{1.41}$ and $2^{1.42}$. In fact, this is how we will define $2^{\sqrt{2}}$. It turns out that there will be exactly one number greater than $2^1, 2^{1.4}, 2^{1.41}, 2^{1.414}, 2^{1.4142}, \ldots$ and less than $2^2, 2^{1.5}, 2^{1.42}, 2^{1.415}, 2^{1.4143}, \ldots$.

And if this sounds like the approximation-by-zooming in we did with the intermediate value theorem, you're right! If x is a rational or decimal approximation to the real number r, then 2^x should be an approximation to 2^r, and as x gets closer to r the approximation should get better. Thus we get the following definition:

Definition 1.16. If r is any real number, and a is a positive real number, we define $a^r = \lim_{x \to r} a^x$ for x varying over the rational numbers. We say that a is the *base* and r is the *exponent*.

Remark 1.17. We can't actually raise a negative real number to an irrational power. The limit would vary over x with even denominator, and a^x is not defined if x has even denominator and $a < 0$.

Proposition 1.18. *The exponential function $f_a(x) = a^x$ is well-defined for any r when $a > 0$, and is continuous on all real numbers. Further, it satisfies the exponential laws:*

- $a^{x+y} = a^x a^y$
- $a^{x-y} = \frac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$.

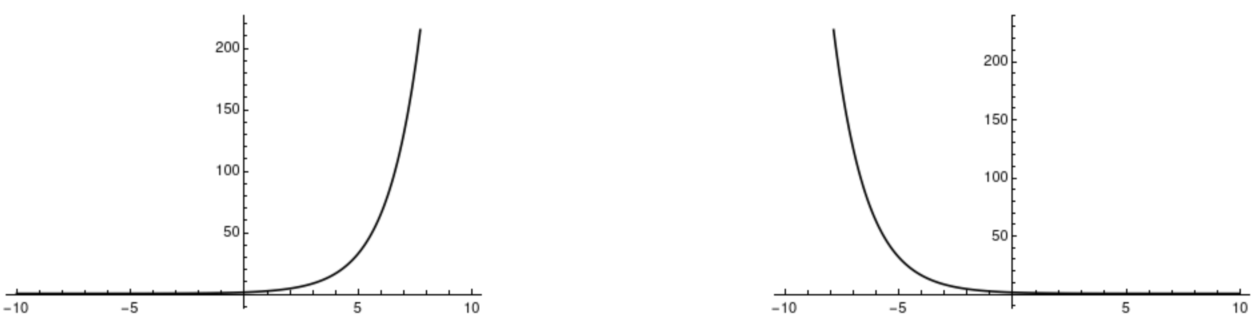


Figure 1.4: The graphs of the exponential functions 2^x and $(1/2)^x$

Proposition 1.19. *If $a > 1$, then $\lim_{x \to +\infty} a^x = +\infty$ and $\lim_{x \to -\infty} a^x = 0$.*

If $0 < a < 1$ then $\lim_{x \to +\infty} a^x = 0$ and $\lim_{x \to -\infty} a^x = +\infty$.

Proof. Both of these can be seen by considering cases where x is an integer. (Or by looking at the graphs.)

When $a > 1$ (say, if $a = 2$, as in figure 1.4), if x is very big then a^x will be very big, and if x is very negative then a^x will be the reciprocal of a very large number, and thus close to 0.

When $0 < a < 1$ (say if $a = 1/2$), if x is very big then a^x will be very close to zero. And if x is very negative then a^x is the reciprocal of a number close to zero, but still positive, and so a^x will be very big. $\square$

There is a number which we will see works much better as a base for the exponential function than any other. This is the number

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

It's possible to prove that this limit exists, but not incredibly easy. It happens that $e \approx 2.71828$. We often write exp for the exponential function with base e; that is, $\exp(x) = e^x$.

Remark 1.20. The number e is also called Euler's number, and was discovered by Jacob Bernoulli in the context of compound interest. (The number was named by Leonhard Euler when he used it for logarithms.)

If your interest rate is r and it's compounded n times a year, then the growth rate per year is $(1 + \frac{r}{n})^n$. If the interest is "compounded continuously," your money grows at a rate equal to the limit of this expression as n goes to $+\infty$—which is e^r.

We'd like to compute the derivative of exp, and also of a^x for a positive real number a. This is a bit difficult to do directly, so instead we're going to cheat.

1.2.2 Logarithms

The exponential function $f(x) = a^x$ is one-to-one, since if $f(x) = f(y)$, then $a^x = a^y$, which means that $a^{x-y} = 1$ and so $x - y = 0$. So a^x must have an inverse function, and we can give it a name.

Definition 1.21. The *logarithmic function with base a*, written $\log_a$, is the inverse function to a^x. It has domain $(0, +\infty)$, and its image is all real numbers.

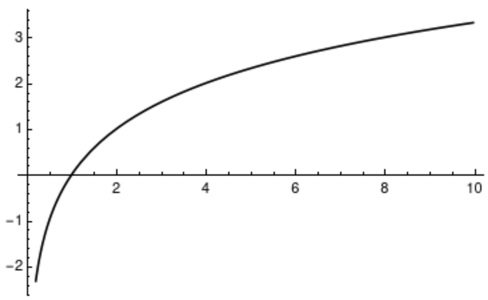
Thus if $a > 0$, we see that $\log_a(a^x) = x$ for every real x,and $a^{\log_a(x)} = x$ for every $x > 0$.

Remark 1.22. Just as there is a natural base e for the exponential, we also most often use e as the base for a logarithm. In this case we call it the *natural logarithm*, denoted ln.

In high school you probably learned that $\log(x)$ means the base-ten logarithm $\log_{10}(x)$. In high school this is definitely true, and it's sometimes true in fields like chemistry, but in other fields it is not true. (Historically, this was more true, since the base-ten logarithm is useful for doing precise calculations by hand; today we use computers instead.)

In computer science, $\log(x)$ usually refers to a base-two logarithm, since binary is very important. In math, $\log(x)$ usually refers to the natural logarithm. In this course I will try to never write $\log(x)$ without specifying a base.

Example 1.23.
- $\log_3(9) = 2$.
- $\log_2(8) = 3$
- $\log_a(1) = 0$ for any $a > 0$.



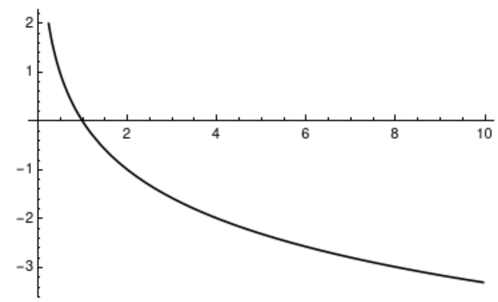


Figure 1.5: The graphs of the exponential functions $\log_2(x)$ and $\log_{1/2}(x)$

Proposition 1.24. *If $a > 1$, then $\lim_{x\to+\infty} \log_a(x) = +\infty$ and $\lim_{x\to 0^+} \log_a(x) = -\infty$.*

If $0 < a < 1$, then $\lim_{x\to+\infty} \log_a(x) = -\infty$ and $\lim_{x\to 0^+} \log_a(x) = +\infty$.

Proof. If x is a large number, this means that we're looking for a number y that will make $a^y = x$ large. Looking at the graph of the exponential function, this implies that y must be large if $a > 1$, then y must be very large, and if $0 < a < 1$, then y must be very negative.

If x is very close to 0, we're looking for a y that will maek $a^y = x$ close to 0. If $a > 1$ this happens when y is very negative; if $0 < a < 1$, this happens when y is very positive.

(We can't compute a limit as $x \to -\infty$ since the logarithm is not defined for negative inputs.) □

The logarithm also has a number of properties corresponding to the exponential laws:

Proposition 1.25. *Our exponential laws imply the following logarithm laws:*
- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$

- $\log_a(x^r) = r\log_a(x)$ *for any real number* r.

Proof.
- We can compute that
$$a^{\log_a(x)+\log_a(y)} = a^{\log_a(x)}a^{\log_a(y)} = xy = a^{\log_a(xy)}.$$
Thus the exponents must be the same, and $\log_a(x) + \log_a(y) = \log_a(xy)$.
- We can compute that
$$a^{\log_a(x)-\log_a(y)} = \frac{a^{\log_a(x)}}{a^{\log_a(y)}} = \frac{x}{y} = a^{\log_a(x/y)}.$$
Thus the exponents must be the same, and $\log_a(x) - \log_a(y) = \log_a(x/y)$.
- We can compute that
$$a^{r\log_a(x)} = \left(a^{\log_a(x)}\right)^r = x^r = a^{\log_a(x^r)}.$$
Thus the exponents must be the same, and $r\log_a(x)\log_a(x^r)$.

□

Example 1.26.
- $\ln(a) + \frac{1}{2}\ln(b) = \ln(a) + \ln(b)^{1/2} = \ln(a\sqrt{b})$.
- Solve $e^{5-3s} = 10$. We have that $5 - 3x = \ln 10$ and so $x = \frac{5-\ln 10}{3}$.

Remark 1.27. These properties are actually historically why the logarithm was originally important. Before calculators, people doing difficult computational work had to work by hand. Adding five digit numbers is much, much easier than multiplying them. So engineers would take the log of the numbers, add them together, and then exponentiate. This was all done with the help of massive books called log tables that would tell you the logarithm of a given number. Slide rules are essentially a way of making the log tables portable; but they were superseded by pocket calculators.

There is one more important logarithmic formula, corresponding to the fourth exponential law from proposition 1.18.

Proposition 1.28 (change of base)**.** *For any positive number* $a \neq 1$*, we have* $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$.

Proof. We use the same approach as in proposition 1.25, but now with the natural logarithm. We see that
$$\exp(\log_a(x)\cdot\ln(a)) = (\exp(\ln(a)))^{\log_a(x)} = a^{\log_a(x)} = x,$$
so $\log_a(x)\cdot\ln(a) = \ln(x)$. □

This allows us to convert logs in any base to logs in another base.

Example 1.29. What is $\log_2 10$? By the change of base formula, we have $\log_2(10) = \frac{\ln 10}{\ln 2}$. $\ln 10 \approx 2.3$ and $\ln 2 \approx .7$, so $\log_2 10 \approx 2.3/.7 \approx 23/7$.

1.3 Derivatives of exponentials and logs

Now we're ready to start computing derivatives. The derivative of exp is hard to do directly, so we start with log.

Proposition 1.30. *The function $f(x) = \log_a(x)$ is differentiable, with derivative $f'(x) = \frac{1}{x}\log_a e$.*

Proof.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\log_a(x+h) - \log_a(x)}{h} \\ &= \lim_{h\to 0} \frac{\log_a((x+h)/x))}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\log_a(1 + \frac{h}{x}) \end{aligned}$$

The next step is maybe a little bit of magic, but we want to simplify the inside of the logarithm, so we define a new variable $y = h/x$. This implies that $h = xy$, and we can replace the limit as $h \to 0$ with a limit as $y \to 0$, so we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{1}{h}\log_a(1 + \frac{h}{x}) \\ &= \lim_{\to 0} \frac{1}{xy}\log_a(1+y) \\ &= \frac{1}{x}\lim_{y\to 0}\log_a\left((1+y)^{1/y}\right) \\ &= \frac{1}{x}\log_a\left(\lim_{y\to 0}(1+y)^{1/y}\right). \end{aligned}$$

But note that this entire limit

$$\lim_{y\to 0}(1+y)^{1/y}$$

doesn't depend on x, or on a, which makes it the same for any logarithmic derivative. So we can just give it a name. In fact, we saw it before, at the end of section 1.2.1. This is the number we call e, and is approximately equal to 2.7.

Later on in the course in section 3.3.2 we'll see why this limit is a reasonable one to write down, beyond the fact that it showed up randomly in this calculation. ☐

Corollary 1.31. *If* $f(x) = \log_a(x)$ *then* $f'(x) = \frac{1}{x \ln a}$.

Proof. By the change of base formula, $\log_a(e) = \frac{\ln(e)}{\ln(a)}$. ☐

Corollary 1.32. $\ln'(x) = \frac{1}{x}$.

Example 1.33.

- Let $f(x) = \log_a(x^3 + 1)$. We know that $\log_a'(x) = \frac{1}{x \ln(a)}$. So by the chain rule, we have

$$\begin{aligned} \frac{d}{dx} \log_a(x^3 + 1) &= \log_a'(x^3 + 1) \cdot (x^3 + 1)' \\ &= \frac{1}{(x^3 + 1)\ln(a)} \cdot 3x^2 = \frac{3x^2}{\ln(a)(x^3 + 1)}. \end{aligned}$$

- Let $g(x) = \ln(\cos(x))$. Then $g'(x) = \frac{1}{\cos(x)} \cdot (-\sin(x)) = -\tan(x)$.

  (In section 1.4 we'll see this gives us an antiderivative for $\tan(x)$.)

Example 1.34. The case where $h(x) = \ln|x|$ is very important. If $x > 0$, then $h(x) = \ln(x)$ and so $h'(x) = \frac{1}{x}$. If $x < 0$ then $h(x) = \ln(-x)$, and then $h'(x) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$. So we get the "new" derivative rule:

$$\ln|x| = \frac{1}{x}.$$

This fact will be really important as we start using the logarithm to compute other derivatives and integrals. It allows us to not worry about whether the function we're taking a logarithm of is positive or not; as long as it is non-zero, we can just throw it into $\ln|x|$ and the derivative will come out the same either way.

We can sometimes use logarithms and implicit differentiation to make difficult differentiation problems easier, just as we use them to simplify difficult arithmetic problems.

Example 1.35 (Power Rule)**.** In calculus 1 we stated the power rule $\frac{d}{dx}x^r = rx^{r-1}$, but only really proved it for the case where r is a positive integer (and even that proof was probably fuzzy). In section 1.1.1 we used the inverse function theorem to prove it if $r = 1/q$ for a positive whole number q, but that really only takes us so far.

But with logarithmic differentiation, we can prove the full version without much work. We compute

$$\begin{aligned} y &= x^r \\ \ln|y| &= r\ln|x| \\ \frac{1}{y}\frac{dy}{dx} &= r\frac{1}{x} \\ \frac{dy}{dx} &= r\frac{y}{x} = r\frac{x^r}{x} = rx^{r-1}. \end{aligned}$$

Remark 1.36. When we compute logarithms of two sides of an equation, we often put both sides in absolute values to ensure the logarithm is actually defined. In this example this isn't really necessary since we know x^r must be positive, but in general it's a good safety measure.

And finally, we can use the logarithmic derivatives to figure out the derivative of the exponential, which we couldn't compute directly.

Proposition 1.37. *If $f(x) = a^x$ for $a > 0$, then f is differentiable and $f'(x) = a^x \ln a$.*

Proof.

$$\begin{aligned} y &= a^x \\ \ln|y| &= x\ln|a| \\ \frac{1}{y}\frac{dy}{dx} &= \ln a \\ \frac{dy}{dx} &= y\ln a = a^x \ln a. \end{aligned}$$

□

Corollary 1.38. $\exp'(x) = \exp(x)$. *(Or we can write $\frac{d}{dx}e^x = e^x$.)*

Example 1.39.
- If $f(x) = e^{\sin(x)}$ then $f'(x) = e^{\sin(x)} \cdot \cos(x)$.
- If $g(x) = 5^{x^2+1}$ then $g'(x) = \ln(5)5^{x^2+1} \cdot 2x$.

We see that generally, $\frac{d}{dx}e^y = e^y \cdot y'$.

Remark 1.40. There's another way to think about this process. You could notice that $a^x = e^{x\ln a}$ so

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln(a)} = e^{x\ln(a)} \cdot \ln(a) = a^x \ln(a).$$

This involves the same basic ideas as the logarithmic approach I presented above, but the implementation is slightly different.

If you like this argument or find it more comfortable, go ahead and use it instead of the logarithmic one. But I think the logarithmic version is much easier when the problems get large and complicated.

Example 1.41. If $h(x) = x^x$ we have to be *very careful*—the obvious approaches don't actually work.

There are two ways you could naively try to answer this problem. The power rule, which assumes the exponent is constant, would give $h'(x)" = "x \cdot x^{x-1}$. The recently-learned exponential rule, which assumes the base is constant, would give $h'(x)" = "x^x \cdot \ln(x)$. Neither of these answers is correct, since the exponent and the base are both variable.

But we can solve this logarithmically:

$$\begin{aligned} y &= x^x \\ \ln|y| &= x \ln|x| \\ \frac{1}{y}\frac{dy}{dx} &= \ln|x| + \frac{x}{x} = \ln|x| + 1 \\ \frac{dy}{dx} &= x^x(\ln|x| + 1). \end{aligned}$$

So $h'(x) = (\ln|x| + 1)x^x$.

(If you prefer the exponential approach, you can write $h(x) = e^{x\ln(x)}$, and thus $h'(x) = e^{x\ln(x)}(\ln(x) + 1) = x^x(\ln(x) + 1)$.)

But there is one extra cool thing I want to point out here. If you pretend the base is constant you get $x^x \cdot \ln(x)$. If you assume the exponent is constant, you get $x \cdot x^{x-1}$, which is the same thing as x^x. If you just add these two formulas together—add up the effect of changing the base, and changing the base—then you get

$$x^x \ln(x) + x^x = x^x(\ln(x) + 1)$$

which is indeed the right answer.

We can also use this logarithmic derivative process to simplify derivatives that we could do in other ways.

Example 1.42. We wish to find the derivative of $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

$$\begin{aligned}
\ln y &= \frac{3}{4}\ln(x) + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{3}{4x} + \frac{2x}{2x^2+2} - \frac{3\cdot 5}{3x+2} \\
\frac{dy}{dx} &= y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right) \\
&= \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right).
\end{aligned}$$

1.4 Integrals involving logarithms and exponentials

Computing these derivative formulas also allows us to do some integrals we couldn't do before.

The obvious rule we've gotten is a rule for integrating exponential functions:

$$\int e^x\,dx = e^x + C.$$

Remark 1.43. We could, if we wanted to, treat this as the definition of e^x: it's the unique (up to a constant) function that's its own derivative. It satisfies the *differential equation* $y' = y$. We'll talk more about this idea in section 3.3.

Example 1.44.

- $\int_0^3 e^x\,dx = e^x|_0^3 = e^3 - 1$.
- $\int_0^{\ln(3)} e^x\,dx = e^x|_0^{\ln(3)} = 3 - 1 = 2$.
- Let's compute $\int e^{3x}\,dx$. We can take $u = 3x$ so $dx = du/3$, and we have

  $$\int e^{3x}\,dx = \int e^u \frac{du}{3} = \frac{1}{3}e^u + C = \frac{1}{3}e^{3x} + C.$$

- We can approach $\int 3^x\,dx$ in a couple of different ways. One approach is to think about the rule that $\frac{d}{dx}3^x = 3^x\ln(3)$, and thus $\int 3^x\,dx = \frac{3^x}{\ln(3)} + C$.

  The other is to do some algebraic "preprocessing". We know that

  $$3^x = (e^{\ln(3)})^x = e^{x\ln(3)}.$$

  Thus we're trying to compute

  $$\int e^{x\ln(3)}\,dx = \frac{1}{\ln(3)}e^{x\ln(3)} + C = \frac{1}{\ln(3)}3^x + C.$$

- Let's compute $\int e^x \cos(1+e^x)\,dx$. Here we take $u = 1+e^x$ so $du = e^x\,dx$, and so we want
$$\int \cos(u)\,du = \sin(u) + C = \sin(1+e^x) + C.$$
- Let's compute $\int x^2 e^{x^3}\,dx$. We can take $u = x^3$ so $du = 3x^2dx$ and we have
$$\int x^2 e^{x^3}\,dx = \int \frac{1}{3}e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C.$$

We learned about the exponential and logarithm, and we learned about the derivative of the exponential and logarithm, so it seems reasonable to think we should now do the integral of the exponential and the logarithm. But that doesn't quite work! The derivative of exp was exp, which allows us to integrate exp. But the derivative of ln was $1/x$; this doesn't actually allow us to integrate ln because we don't have it as the derivative of anything. We will eventually find a way to integrate ln, but that will take tools we don't yet have.

But we do have a much more important integral rule here:
$$\int \frac{1}{x}\,dx = \ln|x| + C.$$
In praactice you can often write $\ln(x)$, but it's safer to write $\ln|x|$; whenever $\ln(x)$ works they mean the same thing, but $\ln|x|$ still works if the denominator of your integrand is negative.

Remark 1.45. Recall that the power rule told us that
$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}$$
whenever $n \neq 1$, but we didn't have a way of integrating x^{-1}. This logarithm rule fills in that gap.

In fact, an alternate path to discover the natural logarithm is to start by trying to find an antiderivative for $\frac{1}{x}$. Some sources will give the definition
$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$
and then define e to be the number that makes this integral equal to 1.

Example 1.46.
- What is the area bounded by $x = 2, x = 3, y = 0$ and $xy = 1$? Drawing the picture, we see we want to compute
$$\int_2^3 \frac{1}{x}\,dx = \ln|x|\Big|_2^3 = \ln(3) - \ln(2) = \ln(3/2) \approx .41.$$

- What is $\int \frac{2x+3}{x^2+3x+5}\,dx$? If we take $u = x^2+3x+5$ then $du = 2x+3dx$, and we have

$$\int \frac{2x+3}{x^2+3x+5}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|x^2+3x+5| + C.$$

- What is $\int \frac{\ln(x)}{x}\,dx$?

  This one looks tricky, but if we take $u = \ln(x)$ so that $du = \frac{1}{x}\,dx$, we see this is

$$\int u\,du = \frac{u^2}{2} + C = \frac{(\ln|x|)^2}{2} + C.$$

Example 1.47. For an even trickier setup, we are finally ready to compute $\int \tan x\,dx$. This isn't obvious at all, but we can see that $\tan x = \frac{\sin x}{\cos x}$; if we take $u = \cos x$, then $du = -\sin(x)\,dx$, and we have

$$\int \tan(x)\,dx = \int -\frac{1}{u}\,du = -\ln|u| + C = -\ln|\cos(x)| + C.$$

Example 1.48 (Recitation challenge)**.** Some integrals here are really truly non-obvious. Suppose we want to compute $\int \frac{dx}{1+e^x}$.

The obvious thing to do is to set $u = e^x$. Then $du = e^x\,dx$ so $dx = du/e^x = du/u$. Then we have

$$\int \frac{dx}{1+e^x} = \int \frac{du/u}{1+u} = \int \frac{du}{u(1+u)}.$$

Using techniques we'll see in a couple weeks, we can work out that $\frac{1}{u(1+u)} = \frac{1}{u} - \frac{1}{u+1}$, and thus the integral is

$$\int \frac{du}{u} - \frac{du}{u+1} = \ln|u| - \ln|u+1| + C = \ln|e^x| - \ln|e^x+1| + C.$$

Alternatively, after some playing around, we can multiply the top and bottom by e^{-x} to get $\int \frac{e^{-x}}{e^{-x}+1}\,dx$. Then we take $u = e^{-x}$ with $du = -e^{-x}\,dx$ so we have

$$\int \frac{e^{-x}}{e^{-x}+1}\,dx = \int \frac{-du}{u+1} = -\ln|u+1| + C = -\ln|e^{-x}+1| + C.$$

It's not at all obvious, but a good exercise, to check that these are the same answer!

1.5 Inverse Trigonometric Functions

We can invert some polynomials, and we can invert exponential functions. The other common category of transcendental functions that we work with is the trigonometric functions, and we'd like to find inverses to these as well.

As a straightforward question, we cannot invert the trigonometric functions because they are all periodic, and thus not one-to-one. For instance, $\sin(0) = \sin(\pi) = \sin(2\pi) = \sin(n\pi)$ for any integer n.

However, sometimes a function is invertible if you restrict its domain enough, to avoid including multiple inputs with the same output. (Often you can achieve this by looking only between two critical points.)

In this section we make canonical domain choices for the trigonometric functions such that they are invertible.

Definition 1.49. If $-1 \leq x \leq 1$, we define:

$$\arcsin(x) = \sin^{-1}(x) = y \text{ where } \sin(y) = x \text{ and } -\pi/2 \leq y \leq \pi/2.$$

The function arcsin has a domain of $[-1, 1]$ and a range of $[-\pi/2, \pi/2]$.

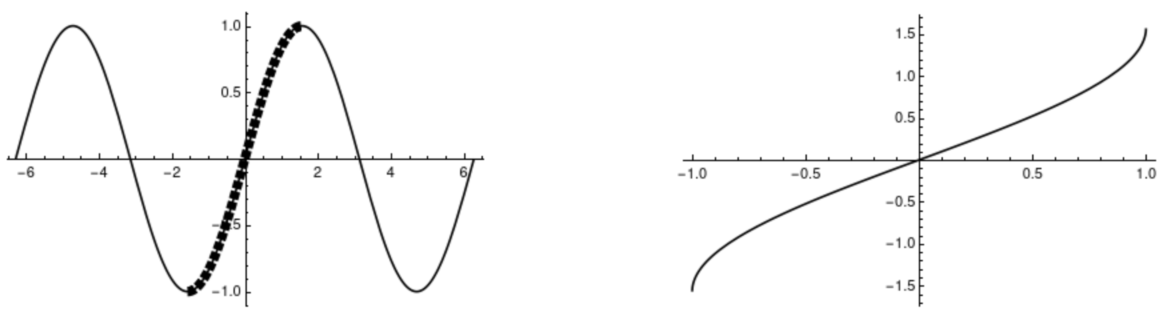


Figure 1.6: Left: A graph of $\sin(x)$ with the restricted domain highlighted
Right: a graph of $\arcsin(y)$

Example 1.50. We can determine that $\arcsin(-\sqrt{3}/2) = -\pi/3$ since $\sin(-\pi/3) = -\sqrt{3}/2$. (Of course, $\sin(5\pi/3) = -\sqrt{3}/2$ as well, but we can ignore this solution because $5\pi/3 > \pi/2$).

With more cleverness, we can calculate $\cos(\arcsin(1/3))$. Suppose $\theta = \arcsin(1/3)$. Then θ is the angle of a triangle with opposite side of lenght 1 and hypotenuse of length 3; using the Pythagorean theorem we determine that the other side has length $\sqrt{8} = 2\sqrt{2}$. Since $\cos(\theta)$ is the length of the adjacent side over the hypotenuse, we have $\cos(\arcsin(1/3)) = 2\sqrt{2}/3$.

We can make similar definitions for inverse cosine and inverse tangent functions. We do have to be careful about the precise domains and images.

Definition 1.51. If $-1 \leq x \leq 1$, we define

$$\arccos(x) = \cos^{-1}(x) = y \text{ where } \cos(y) = x \text{ and } 0 \leq y \leq \pi.$$

This function has domain $[-1, 1]$ and range $[0, \pi]$.

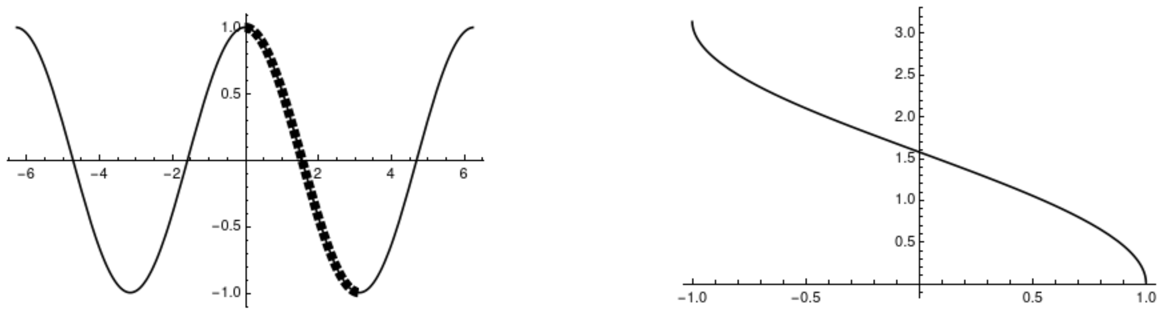


Figure 1.7: Left: A graph of $\cos(x)$ with the restricted domain highlighted
Right: a graph of $\arccos(y)$

Definition 1.52. If x is a real number, we define:

$$\arctan(x) = \tan^{-1}(x) = y \text{ where } \tan(y) = x \text{ and } -\pi/2 < y < \pi/2.$$

This function has domain $(-\infty, +\infty)$ and image $(-\pi/2, \pi/2)$. (Note the strict inequalities $<$ here, rather than the $\leq$ we used for sine and cosine.)

Because the domain here is infinite, we want to think about the limits of this function as well. We know that when x is close to $\pi/2$ then $\tan(x)$ is very large; turning this around, we see that $\lim_{x\to+\infty} \arctan(x) = \pi/2$. similarly $\lim_{x\to-\infty} \arctan(x) = -\pi/2$.

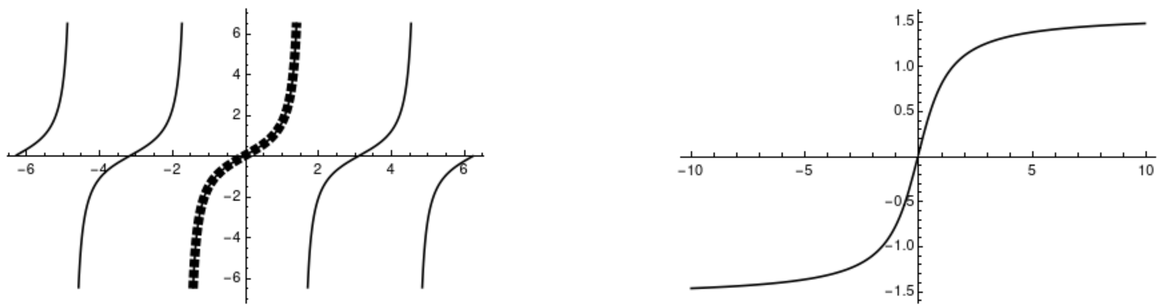


Figure 1.8: Left: A graph of $\tan(x)$ with the restricted domain highlighted
Right: a graph of $\arctan(y)$

The trigonometric functions sin and cos and tan are all differentiable, so by the Inverse Function Theorem 1.13, so are arcsin and arccos and arctan, at least most of the time.

Proposition 1.53. *We have the following derivative formulas:*

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}.$$

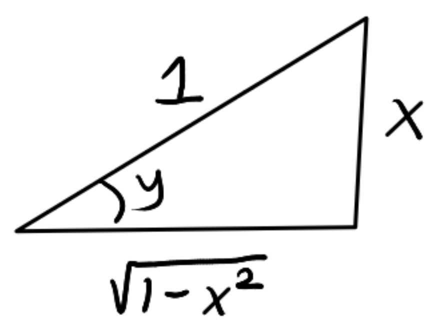
Proof. There are two approaches to proving these facts. One involves trigonometric identities, and the other involves thinking about triangles. They both involve implicit differentiation.

Suppose $y = \arcsin(x)$. Then $\sin(y) = x$ and thus $\cos(y)\frac{dy}{dx} = 1$. Then we have $\frac{dy}{dx} = \frac{1}{\cos(y)}$.

From here, we have two different approaches. One is to note that $\cos(y) = \sqrt{1-\sin^2(y)}$ by the Pythagorean trigonometric identity, and since $y = \arcsin(x)$ we know that $\sin(y) = x$. Thus $\cos(y) = \sqrt{1-x^2}$, and so

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}.$$

I find it easier to think about a different approach, though. If $y = \arcsin(x)$, then y is the angle of a triangle where the opposite side has length x and the hypotenuse has length 1. Then by the Pythagorean theorem, the third side has length $\sqrt{1-x^2}$, so



$$\cos(y) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}.$$

Note we got the same answer both ways, and they both involved basically the same facts; the identity $\sin^2(y) + \cos^2(y) = 1$ holds precisely because of the triangle argument. Either way you want to think of it is fine with me.

We can do the same with $\arccos(x)$. We set $\cos(y) = x$, so

$$\frac{dy}{dx} = \frac{-1}{\sin(y)} = -\frac{1}{\sqrt{1-x^2}}.$$

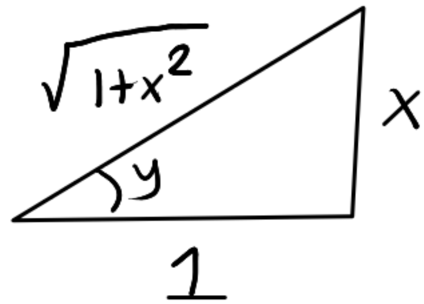
Working out the derivative of arctan is slightly trickier. We set $\tan(y) = x$ so $\sec^2(y)\frac{dy}{dx} = 1$, and thus we have $\frac{dy}{dx} = \cos^2(y)$. We again have two approaches:

First, we can use the identity $1 + \tan^2(y) = \sec^2(y)$, which gives us

$$\cos^2(y) = \frac{1}{\sec^2(y)} = \frac{1}{1+\tan^2(y)} = \frac{1}{1+x^2}$$

since $\tan(y) = x$.

Second, we can see that y is the angle of a triangle with opposite side x and adjacent side 1, and hence hypotenuse $\sqrt{1+x^2}$. Then $\cos(y) = \frac{1}{\sqrt{1+x^2}}$ and so $\arctan'(x) = \cos^2(y) = \frac{1}{1+x^2}$.



□

Example 1.54. • What is $\arcsin'(3/4)$?

We know that $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$, so $\arcsin'(3/4)$ is $\frac{1}{\sqrt{1-(3/4)^2}} = \frac{1}{\sqrt{7/16}}$.

- What is $\frac{d}{dx}\arctan(e^x)$?

  Since $\arctan'(x) = \frac{1}{1+x^2}$, we have

$$\frac{d}{dx}\arctan(e^x) = \arctan'(e^x) \cdot (e^x)' = \frac{1}{1+(e^x)^2} \cdot e^x = \frac{e^x}{1+e^{2x}}.$$

- What is $\frac{d}{dx}\arccos(x^2+2x+3)$?

  We get $\dfrac{-1}{\sqrt{1-(x^2+2x+3)^2}} \cdot (2x+2)$.

Remark 1.55. There are also some other derivative formulas that almost no one cares about.

$$\frac{d}{dx}\operatorname{arccot}(x) = \frac{-1}{1+x^2} \qquad \frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\operatorname{arccsc}(x) = \frac{-1}{x\sqrt{x^2-1}}.$$

It's actually a little annoying to define the ranges of these functions, and we mostly avoid using them, but I list the formulas here for completeness.

1.5.1 Integrals with Inverse Trig Functions

These new derivative formulas give us new integral formulas. In practice we only really use two:

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$$
$$\int \frac{dx}{1+x^2} = \arctan(x) + C.$$

The second in particular comes up really often in certain integrals.

Example 1.56. Let's compute $\int \frac{dx}{x^2+4}$.

There are a couple of ways to do this, but the most straightforward is to try to massage it into something with a 1 on the bottom. So we can observe

$$\frac{1}{x^2+4} = \frac{1/4}{x^2/4+1} = \frac{1}{4}\frac{1}{(x/2)^2+1}.$$

So we can take $u = x/2$ with $du = dx/2$, and get

$$\begin{aligned}
\int \frac{dx}{x^2+4} &= \int \frac{1}{4}\frac{1}{(x/2)^2+1}\,dx \\
&= \int \frac{1}{4}\frac{1}{u^2+1}\cdot 2\,du \\
&= \frac{1}{2}\int \frac{1}{u^2+1}\,du \\
&= \frac{1}{2}\arctan(u) + C = \frac{1}{2}\arctan(x/2) + C.
\end{aligned}$$

Example 1.57. Let's compute $\int \frac{x}{\sqrt{1-x^4}}\,dx$.

If the denominator were just $\sqrt{1-x^2}$ this would be a simple u-substitution. But here we need something a bit more.

We can take $u = x^2$ so that $du = 2x\,dx$. Then

$$\begin{aligned}
\int \frac{x}{\sqrt{1-x^4}}\,dx &= \int \frac{x}{\sqrt{1-u^2}}\frac{du}{2x} \\
&= \int \frac{1}{2}\frac{1}{\sqrt{1-u^2}}\,du \\
&= \frac{1}{2}\arcsin(u) + C = \frac{1}{2}\arcsin(x^2) + C.
\end{aligned}$$

Some sources will list the following integral rules for simplicity:

$$\begin{aligned}
\int \frac{dx}{a^2+x^2} &= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \\
\int \frac{dx}{\sqrt{a^2-x^2}} &= \arcsin\left(\frac{x}{|a|}\right) + C.
\end{aligned}$$

There's one more technique we can find useful here, called *completing the square*. This is something almost everyone learned in high school, and then promptly forgot a week later when it was supplanted as a tool for solving equations by the quadratic formula.

Example 1.58. Let's compute $\int \frac{dx}{x^2+2x+5}$.

We want the bottom to look like u^2+1. So first we complete the square to get rid of the $2x$ term. We want to find a number a so that x^2+2x+a is a perfect square; we see that $x^2+2x+1=(x+1)^2$. So we have

$$\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x^2+2x+1)+4} = \int \frac{dx}{(x+1)^2+4}$$
$$= \frac{1}{4}\frac{dx}{((x+1)/2)^2+1}.$$

Now we set $u=(x+1)/2$ so $du = dx/2$, and we get

$$\int \frac{dx}{x^2+2x+5} = \int \frac{1}{4}\frac{2du}{u^2+1} = \frac{1}{2}\int \frac{du}{u^2+1}$$
$$= \frac{1}{2}\arctan(u)+C = \frac{1}{2}\arctan((x+1)/2)+C.$$

1.5.2 A note on hyperbolic trigonometric functions

There are some trigonometric-like functions called the hyperbolic trig functions. The basic formulas are

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

and then $\tanh, \coth, \operatorname{sech}, \operatorname{csch}$ are defined as they are for regular trig functions. You can wrok out the derivatives of these functions, and get what you'd maybe expect from the names:

$$\frac{d}{dx}\sinh(x) = \cosh(x) \qquad \frac{d}{dx}\cosh(x) = -\sinh(x) \qquad \frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$$

and so on. We can also define the inverse hyperbolic trigonometric functions and get some familiar-looking formulas, which are occasionally useful:

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}.$$

However, none of these formulas are useful often enough for me to actually want to teach them. It's enough to know that these formulas do exist, and you can look them up if you need to.

As a final note, these definitions don't *look* at all like the regular trig functions, so it's surprising that all the other results work out the same. However, we'll see at the very end of the class that if you allow the imaginary number $i=\sqrt{-1}$, then we can take

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}.$$

Now we see the relationship between trig and hyperbolic trig functions: we get the hyperbolic trig functions by just ignoring the i terms in these formulas.

1.6 L'Hospital's Rule

We're going to finish by talking about how we can compute limits of transcendental functions like ln and exp. Some of these turn out to be easy:

Example 1.59.

$$\lim_{x\to 1} \frac{\ln(x)^{\nearrow 0}}{x_{\searrow 1}} = \frac{0}{1} = 0.$$

But some of them do not. If we want to compute

$$\lim_{x\to 1} \frac{\ln(x)^{\nearrow 0}}{x-1_{\searrow 0}}$$

we need some more tools.

In general, we only have a problem if our limit is an "indeterminate form", like "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$". There is a very powerful tool we can develop that only works for indeterminate forms; but since indeterminate forms are the only tricky ones, that limitation isn't a real problem.

Theorem 1.60 (L'Hospital's Rule)**.** *Suppose f and g are differentiable, and $g'(x) \neq 0$ near a, except possibly at a. Suppose either $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$. (In other words, the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form). Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right exists.

Remark 1.61. This rule is named after Guillame François Antoine, the Marquis de l'Hospital. (It was discovered by Johann Bernoulli, the brother of the Jacob Bernoulli who discovered e.) Guillame spelled his name "l'Hospital", but later French orthographic reforms shifted the spelling to "l'Hôpital". I follow the more traditional spelling that the Marquis himself used, but you'll see both used interchangeably.

Informal sketch. Let's assume that $f(a) = g(a) = 0$ and $g'(a) \neq 0$. Then by linear approximation we know that $f(x) \approx f(a) + f'(a)(x-a)$, and similarly $g(a) \approx g(a) + g'(a)(x-a)$. Then we have

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)} = \frac{f'(a)(x-a)}{g'(a)(x-a)}$$
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}.$$

□

Proof. Now let's prove this a bit more rigorously, but still staying in the case where $f(a) = g(a) = 0$, $g'(a) \neq 0$, and f' and g' are continuous at a.

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} \\
&= \lim_{x\to a} \frac{(f(x)-f(a))(x-a)}{(g(x)-g(a))(x-a)} \\
&= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} \\
&= \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.
\end{aligned}$$

$\square$

Example 1.62.

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2-4x+3^{\nearrow 0}}{x^2-2x-3_{\searrow 0}} &=^{\text{L'H}} \lim_{x\to 3} \frac{2x-4}{2x-2} = \frac{2}{4} = \frac{1}{2}. \\
\lim_{x\to 0} \frac{1-\cos(x)^{\nearrow 0}}{\sin(x)_{\searrow 0}} &=^{\text{L'H}} \lim_{x\to 0} \frac{\sin(x)}{\cos(x)} = \frac{0}{1} = 0. \\
\lim_{x\to 1} \frac{\ln x^{\nearrow 0}}{x-1_{\searrow 0}} &=^{\text{L'H}} \lim_{x\to 0} \frac{1/x}{1} = 1.
\end{aligned}$$

Sometimes we have to apply L'Hôpital's rule more than once to get the results we want.

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan x - x^{\nearrow 0}}{x^3_{\searrow 0}} &=^{\text{L'H}} \lim_{x\to 0} \frac{\sec^2(x)-1^{\nearrow 0}}{3x^2_{\searrow 0}} \\
&=^{\text{L'H}} \lim_{x\to 0} \frac{2\sec^2(x)\tan(x)}{6x} = \lim_{x\to 0} \sec^2(x) \lim_{x\to 0} \frac{\tan x}{3x} = 1 \cdot \lim_{x\to 0} \frac{\tan(x)^{\nearrow 0}}{3x_{\searrow 0}} \\
&=^{\text{L'H}} \lim_{x\to 0} \frac{\sec^2(x)}{3} = \frac{1}{3}.
\end{aligned}$$

$$\lim_{x\to 0} \frac{e^x - 1 - x^{\nearrow 0}}{x^2_{\searrow 0}} =^{\text{L'H}} \lim_{x\to 0} \frac{e^x - 1^{\nearrow 0}}{2x_{\searrow 0}} =^{\text{L'H}} \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}.$$

We can also use L'Hôpital's rule to evaluate limits at infinity.

Example 1.63.

$$\lim_{x\to\pm\infty} \frac{x^2+5x+3^{\nearrow\infty}}{x^2+7x-2_{\searrow\infty}} =^{\text{L'H}} \lim_{x\to\pm\infty} \frac{2x+5^{\nearrow\infty}}{2x+7_{\searrow\infty}}$$

$$=^{\text{L'H}} \lim_{x\to\pm\infty} \frac{2}{2} = 1.$$

$$\lim_{x\to+\infty} \frac{\ln(x)^{\nearrow\infty}}{x_{\searrow\infty}} =^{\text{L'H}} \lim_{x\to+\infty} \frac{1/x}{1} = 0.$$

$$\lim_{x\to+\infty} \frac{e^{x\nearrow\infty}}{x_{\searrow\infty}} =^{\text{L'H}} \lim_{x\to+\infty} \frac{e^x}{1} = +\infty.$$

In fact, it's not too hard to see, using L'Hôpital's Rule, that $\lim_{x\to+\infty} \frac{e^x}{x^n} = +\infty$ and $\lim_{x\to+\infty} \frac{\ln(x)}{x^n} = 0$.

We sometimes say that $\ln(x)$ grows slower than any possible polynomial, and e^x grows faster.

Remember that L'Hôpital's rule only applies if we start with an indeterminate form.

Example 1.64.

$$\lim_{x\to\pi} \frac{\sin(x)^{\nearrow 0}}{1-\cos(x)_{\searrow 2}} \neq \frac{\cos(x)}{\sin(x)} = \pm\infty$$

$$\lim_{x\to\pi} \frac{\sin(x)^{\nearrow 0}}{1-\cos(x)_{\searrow 2}} = \frac{0}{1-(-1)} = 0.$$

A more dangerous example:

$$\lim_{x\to 0} \frac{e^x-1-x^{\nearrow 0}}{x^3_{\searrow 0}} =^{\text{L'H}} \lim_{x\to 0} \frac{e^x-1^{\nearrow 0}}{3x^2_{\searrow 0}} =^{\text{L'H}} \lim_{x\to 0} \frac{e^{x\nearrow 0}}{6x_{\searrow 0}}$$

You might think we should use L'Hôpital's rule again here; that would give $\lim_{x\to 0} \frac{e^x}{6} = 1/6$. But the top goes to 1 and the bottom goes to 0, so this is not an indeterminate form! The true limit is $\pm\infty$.

And sometimes L'Hôpital's rule doesn't alwasy work the way we'd like it to, just "because it doesn't."

Example 1.65.

$$\lim_{x\to+\infty} \frac{x^{\nearrow 0}}{\sqrt{x^2+1}_{\searrow 0}} =^{\text{L'H}} \lim_{x\to+\infty} \frac{1}{\frac{x}{\sqrt{x^2+1}}} = \lim_{x\to+\infty} \frac{\sqrt{x^2+1}^{\nearrow 0}}{x_{\searrow 0}}$$

$$=^{\text{L'H}} \lim_{x\to+\infty} \frac{\frac{x}{\sqrt{x^2+1}}}{1} = \lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}}.$$

But here if we're clever we can observe that if the limit exists, then

$$\left(\lim_{x\to+\infty}\frac{x}{\sqrt{x^2+1}}\right)^2 = \lim_{x\to+\infty}\frac{x^2}{x^2+1} = 1$$
$$\lim_{x\to\pm\infty}\frac{x}{\sqrt{x^2+1}} = \pm 1.$$

Alternatively, we can just fall back on our techniques from Calculus 1:

$$\lim_{x\to+\infty}\frac{x}{\sqrt{x^2+1}} = \lim_{x\to+\infty}\frac{1}{\sqrt{1+1/x^2}} = \frac{1}{\sqrt{1+0}} = 1.$$

We can often use L'Hôpital's rule to compute limits of other indeterminate forms with a bit of cleverness. Recall the "minor" indeterminate forms are $1^\infty, \infty-\infty, 0^0, \infty^0, 0\cdot\infty$. Products can obviously be rewritten as quotients, and sums or differences can often be combined into something by collecting common denominators. Exponents can be turned into ratios by means of logarithms.

Example 1.66. Our first example is $\lim_{x\to\pi/2}\sec(x)-\tan(x)$, which looks like "$\infty-\infty$". This doesn't require logarithms, but we need to do some pre-processing before we can use L'Hospital's Rule.

$$\begin{aligned}\lim_{x\to\pi/2}\sec(x)-\tan(x) &= \lim_{x\to\pi/2}\left(\frac{1}{\cos(x)}-\frac{\sin(x)}{\cos(x)}\right)\\ &= \lim_{x\to\pi/2}\frac{1-\sin(x)^{\nearrow 0}}{\cos(x)_{\searrow 0}}\\ &=^{\text{L'H}} \lim_{x\to\pi/2}\frac{-\cos(x)^{\nearrow 0}}{-\sin(x)_{\searrow 1}} = \frac{0}{1} = 0.\end{aligned}$$

Example 1.67. The example $\lim_{x\to 0}\cot(2x)\sin(6x)$ looks like "$\infty\cdot 0$". Again, we don't need logarithms, but we do need to do some reorganization before we can use L'Hospital's.

$$\begin{aligned}\lim_{x\to 0}\cot(2x)\sin(6x) &= \lim_{x\to 0}\frac{\sin(6x)\cos(2x)}{\sin(2x)} = 1\cdot\lim_{x\to 0}\frac{\sin(6x)^{\nearrow 0}}{\sin(2x)_{\searrow 0}}\\ &= \lim_{x\to 0}\frac{6\cos(6x)^{\nearrow 6}}{2\cos(2x)_{\searrow 2}} = 3.\end{aligned}$$

Example 1.68. Now let's compute $\lim_{x\to 1}x^{1/(1-x)}$, which looks like "1^∞". Since we have a complicated exponent, this begs for logarithms.

$$\begin{aligned}
y &= x^{1/(1-x)} \\
\ln(y) &= \frac{1}{1-x}\ln(x) = \frac{\ln(x)}{1-x} \\
\lim_{x\to 1}\ln(y) &= \lim_{x\to 1}\frac{\ln(x)^{\nearrow 0}}{1-x_{\searrow 0}} =^{\text{L'H}} \lim_{x\to 1}\frac{1/x}{-1} = -1 \\
\lim_{x\to 1} y &= e^{-1} = \frac{1}{e}.
\end{aligned}$$

Example 1.69. Let's compute $\lim_{x\to+\infty} x^{1/x}$, which looks like "$\infty^0$". Again, we have a complicated exponent, so again we use logarithms.

$$\begin{aligned}
\ln(y) &= \frac{1}{x}\ln(x) = \frac{\ln(x)}{x} \\
\lim_{x\to+\infty}\ln(y) &= \lim_{x\to+\infty}\frac{\ln(x)^{\nearrow\infty}}{x_{\searrow\infty}} =^{\text{L'H}} \lim_{x\to+\infty}\frac{1/x}{1} = 0 \\
\lim_{x\to+\infty} y &= e^0 = 1.
\end{aligned}$$

Example 1.70. Finally, let's compute $\lim_{x\to 0^+} x^{\frac{1}{\ln(x)-1}}$, which looks like "$0^0$".

Again we use logarithms.

$$\begin{aligned}
\ln(y) &= \frac{1}{\ln(x)-1}\ln(x) = \frac{\ln(x)}{\ln(x)-1} \\
\lim_{x\to 0^+}\ln(y) &= \lim_{x\to 0^+}\frac{\ln(x)^{\nearrow\infty}}{\ln(x)-1_{\searrow\infty}} =^{\text{L'H}} \lim_{x\to 0^+}\frac{1/x}{1/x} = 1 \\
\lim_{x\to 0^+} y &= e^1 = e.
\end{aligned}$$

Remark 1.71. The value of 0^0 computed directly is a good question to start bar fights at math conferences. In most non-calculus contexts, the correct answer is 1, but in calculus it depends on exactly which limit you're computing—which is exactly the definition of an indeterminate form.

We can come back to this idea in multivariable calculus: we can say that

$$\lim_{(x,y)\to(0,0)} x^y$$

is indeterminate, and depends on exactly how x and y are related. In example 1.70 we take $y = \frac{1}{\ln(x)-1}$, and then the limit becomes quite determinate—and definitely not equal to 1.