2 Advanced Integration Techniques

In calculus 1 we learned the basics of calculating integrals; in sections 1.4 and 1.5.1 we found some additional formulas that enable us to integrate more functions. But there are still plenty of relatively simple integrals we don't have a way to compute, like

$$\int x \sin(x)\, dx \qquad \int \sin^2(x)\, dx \qquad \int \sqrt{1-x^2}\, dx \qquad \int \frac{1}{x^2-1}\, dx \qquad \int e^{-x^2}\, dx.$$

In this section we'll study some more advanced techniques for finding integrals that will let us handle all of the above questions.

The important skill here isn't simply being able to come up with integral formulas; there are plenty of easy-to-use computer tools that will let you do that. Instead, this material has two goals. First, understanding how some basic, common integrals work makes it easier to intuitively understand applications in other fields. Second and related, I hope learning these techniques illuminates a bit of *why* integrals behave the way they do.

2.1 Integration by Parts

How do we integrate a product of two functions? Sometimes this is easy: if one function is constant, we can just pull it out of the integral; and if one piece is the derivative of the other, we can use u-substitution. But in general we don't have a good way to handle the product of two unrelated functions.

We observed earlier that integrals like addition and scalar multiplication, but don't work well with function multiplication. However, we had a straightforward multiplication rule for derivatives: remember that $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$. We said that every derivative rule gives us an integral rule; can we use this rule somehow to get a product rule for integrals?

Well, what happens if we integrate both sides? It follows from the Fundamental Theorem of Calculus that $\int \frac{d}{dx} f(x)g(x), dx = f(x)g(x) + C$. Thus

$$\int f'(x)g(x) + f(x)g'(x)\, dx = f(x)g(x)$$
$$\int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx.$$

Thus if we have an integral we can write as $\int f(x)g'(x)\, dx$, and and we know $\int f'(x)g(x)\, dx$, we can find an antiderivative. This is also sometimes written for bookkeeping as

$$\int u\, dv = uv - \int v\, du.$$

Remark 2.1. I'll mostly compute antiderivatives rather than definite integrals, since you can definitely compute a definite integral if you can find the antiderivative. But we *can* state our result in terms of definite integrals as well:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)\,dx.$$

Example 2.2. Find $\int xe^x\,dx$. We want to write this as a product of two functions, one of which we can easily differentiate and the other we can easily integrate. So let $u = x$ and $dv = e^x\,dx$; then we have $du = 1\,dx$ and $v = e^x$. Thus by integration by parts, we have

$$\begin{aligned}\int u\,dv &= uv - \int v\,du \\ \int xe^x\,dx &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C = (x-1)e^x + C.\end{aligned}$$

Indeed, we can check that $((x-1)e^x)' = e^x + (x-1)e^x = xe^x$.

Example 2.3. Let's find $\int_0^\pi x\cos(x)\,dx$. Again, we note that x becomes simpler when we take a derivative, so we set $u = x, dv = \cos(x)$, and then $du = dx$ and $v = \sin(x)\,dx$. We get

$$\begin{aligned}\int_0^\pi x\cos(x)\,dx &= x\sin(x)\Big|_0^\pi - \int_0^\pi \sin(x)\,dx \\ &= \pi\sin(\pi) - 0\sin(0) - \left(-\cos(x)\Big|_0^\pi\right) \\ &= \cos(\pi) - \cos(0) = -2.\end{aligned}$$

In addition to the "obvious" applications where we're integrating a product, we can sometimes use this method when we have a function with no obvious integral, but whose derivative is much simpler.

Example 2.4 (antiderivative of $\ln(x)$)**.** Consider $\int \ln(x)\,dx$. This doesn't look like integration by parts should help, since it's not a product. But $\ln(x)$ gets much easier to deal with after we take a derivative, so we can try it out. Let $u = \ln(x), dv = 1\,dx$, and thus $du = \frac{1}{x}\,dx$ and $v = x$. We have

$$\begin{aligned}\int u\,dv &= uv - \int v\,du \\ \int \ln(x)\,dx &= x\ln(x) - \int x\frac{1}{x}\,dx \\ &= x\ln(x) - \int 1\,dx = x\ln(x) - x + C.\end{aligned}$$

Indeed, we can check that $(x\ln(x) - x)' = \ln(x) + \frac{x}{x} - x = \ln(x)$.

Integration by parts takes our original integral and replaces it with a new integral we have to do. Sometimes that new integral is also an integral of a product, so we have to repeat the integration by parts process.

Example 2.5. Consider $\int x^2 \cos(x)\, dx$. We can take $u = x^2, dv = \cos(x)\, dx$, so $du = 2x\, dx, v = \sin(x)$. Then

$$\int x^2 \cos(x)\, dx = x^2 \sin(x) - 2 \int x \sin(x)\, dx.$$

We don't really know $\int x \sin(x)\, dx$ either, but we can take $u = x, dv = \sin(x)\, dx$ so $du = dx, v = -\cos(x)$. Then

$$\begin{aligned}
\int x \sin(x)\, dx &= -x \cos(x) - \int -\cos(x)\, dx \\
&= -x\cos(x) + \sin(x) + C \\
\int x^2 \cos(x)\, dx &= x^2 \sin(x) - 2\left(-x\cos(x) + \sin(x) + C\right) \\
&= x^2 \sin(x) + 2x \cos(x) - 2\sin(x) + C.
\end{aligned}$$

We check our work by taking a derivative, and get

$$2x \sin(x) + x^2 \cos(x) + 2\cos(x) - 2x\sin(x) - 2\cos(x) = x^2 \cos(x).$$

Example 2.6. Sometimes repeating the integration-by-parts process leads to a repeating cycle; surprisingly, this can still give us an answer.

Consider $\int \sin(x) e^x\, dx$. This is clearly a product, and neither of these becomes particularly simpler or more complex by integrating or differentiating. Still, let's give it a try.

$$\int \sin(x) e^x\, dx = \sin(x) e^x - \int \cos(x) e^x\, dx$$

This doesn't seem to help, because the new integral isn't any easier than the old. Let's keep going anyway.

$$\int \cos(x) e^x\, dx = \cos(x) e^x - \int (-\sin(x) e^x)\, dx$$

and this last integral is the same as the one we started with. This doesn't look promising, but it actually works out fine.

$$\begin{aligned}
\int \sin(x)e^x \, dx &= \sin(x)e^x - \left(\cos(x)e^x + \int \sin(x)e^x \, dx\right) \\
&= \sin(x)e^x - \cos(x)e^x - \int \sin(x)e^x \, dx \\
2\int \sin(x)e^x \, dx &= \sin(x)e^x - \cos(x)e^x + C \\
\int \sin(x)e^x &= \frac{e^x}{2}(\sin(x) - \cos(x)) + C.
\end{aligned}$$

Example 2.7. Consider $\int \sin(x)\cos(x)\, dx$. This is a product, so we can use integration by parts.

$$\begin{aligned}
\int \sin(x)\cos(x)\, dx &= \sin^2(x) - \int \sin(x)\cos(x)\, dx \\
2\int \sin(x)\cos(x)\, dx &= \sin^2(x) + C \\
\int \sin(x)\cos(x)\, dx &= \frac{1}{2}\sin^2(x) + C.
\end{aligned}$$

However, here we could have done something much easier: if $u = \sin(x)$ then $du = \cos(x)\, dx$ and we get

$$\int \sin(x)\cos(x)\, dx = \int u\, du = u^2/2 + C = \frac{1}{2}\sin^2(x) + C.$$

Example 2.8. Finally, consider $\int \cos^2(x)\, dx$. There's no useful u-substitution, but we can look at the integrand as $\cos(x) \cdot \cos(x)$ and use integration by parts. We get

$$\begin{aligned}
\int \cos(x) \cdot \cos(x)\, dx &= \sin(x)\cos(x) - \int -\sin(x) \cdot \sin(x)\, dx \\
&= \sin(x)\cos(x) + \int \sin(x) \cdot \sin(x)\, dx \\
&= \sin(x)\cos(x) + \left(-\sin(x)\cos(x) + \int \cos(x) \cdot \cos(x)\, dx\right) \\
&= 0 + \int \cos(x) \cdot \cos(x)\, dx.
\end{aligned}$$

This is clearly *true* but not at all useful to us. We need to develop some other tools if we want to compute this successfully. So in the next section we'll study how to do trigonometric integrals that aren't already on our formula lists.

2.2 Trigonometric Integrals

2.2.1 Integrals of Trigonometric Functions

So far we've found antiderivatives for a number of trigonometric functions, including sin, cos, tan. Here we study some trigonometric identities that allow us to integrate more difficult functions.

I often say that there are really only two or three trigonometric identities you need to know.

- $\sin^2(x) + \cos^2(x) = 1$.
- $\sin^2(x) = \dfrac{1 - \cos 2x}{2}$.
- $\cos^2(x) = \dfrac{1 + \cos 2x}{2}$.

The second and third are called the "double angle" formulas. I'll call the first the "circle identity" or "pythagorean identity" but that's not a standard name.

Remark 2.9. I say these are the only identities you need to know. However, there are many other identities you can derive from these. As a warning, many problems involving trigonometric functions have multiple solutions which all appear to be different, but are actually the same.

Corollary 2.10. *(a)* $1 + \tan^2(x) = \sec^2(x)$.

(b) $1 + \cot^2(x) = \csc^2(x)$.

We will use these identities to massage our integrals into something doable. For integrals involving powers of sin and cos, our general strategy is to write our integrand as a sum of things with either sin or cos being a first power, and then substituting u for the function with a higher power.

General rule: if you have an odd number of sines or of cosines, you can use the circle identity to get just one sine or just one cosine, which will be your du. If you have an even number of both, use the double-angle formula on all of them to cut the total number in half, until you have an odd number of at least one, then use the circle identity as before.

Example 2.11. If we integrate an even power of sin or cos, we must use the double angle formulas to reduce it.

$$\begin{aligned}
\int \cos^2(x)\,dx &= \frac{1}{2}\int 1+\cos(2x)\,dx \\
&= \frac{1}{2}\int (1+\cos(u))\frac{du}{2} \\
&= \frac{1}{4}(u+\sin(u)) = \frac{x}{2}+\frac{\sin(2x)}{4}+C
\end{aligned}$$

(where $u = 2x, du = 2dx$).

Example 2.12. If we have an odd power of sin or cos, we can use the circle identity to reduce it.

$$\begin{aligned}
\int \sin^3(x)\,dx &= \int \sin(x)(1-\cos^2(x))\,dx \\
&= \int \sin(x)-\sin(x)\cos^2(x)\,dx = \int \sin(x)\,dx - \int \sin(x)\cos^2(x)\,dx \\
&= -\cos(x) + \int u^2\,du = \frac{1}{3}u^3 - \cos(x) = \frac{1}{3}\cos^3(x)-\cos(x)+C
\end{aligned}$$

(where $u = \cos(x), du = -\sin(x)dx$).

Example 2.13.

$$\begin{aligned}
\int \cos^8(x)\sin^3(x)\,dx &= \int \cos^8(x)(1-\cos^2(x))\sin(x)\,dx \\
&= \int \cos^8(x)\sin(x)-\cos^{10}(x)\sin(x)\,dx \\
&= \int u^{10}-u^8\,du = \frac{u^{11}}{11}-\frac{u^9}{9}+C = \frac{\cos^{11}(x)}{11}-\frac{\cos^9(x)}{9}+C
\end{aligned}$$

where $u = \cos(x), du = -\sin(x)\,dx$.

Example 2.14. The trickiest case is when you have even powers of both sine and cosine. Mostly this just leads to a large pile of unpleasant algebra.

$$\begin{aligned}
\int \sin^2(x)\cos^2(x)\,dx &= \int \frac{1}{2}(1-\cos 2x)\,\frac{1}{2}(1+\cos(2x))\,dx \\
&= \frac{1}{4}\int 1-\cos^2 2x\,dx \\
&= \frac{1}{4}\int 1-\frac{1}{2}(1+\cos 4x)\,dx \\
&= \frac{1}{4}\left(x-\frac{1}{2}(x+\frac{\sin 4x}{4})\right) \\
&= \frac{3x}{8}-\frac{\sin(4x)}{8}+C.
\end{aligned}$$

Integrals with secant and tangent work slightly differently. We try to use the fact that $1+\tan^2(x) = \sec^2(x)$ to rewrite the expression so that either there's exactly one tangent, or exactly two secants, and then a u substitution will work.

Example 2.15. If we have an even number of secants we use our identity to rewrite so we have exactly two secants. We will wind up setting $u = \tan\theta$ and $du = \sec^2\theta\, d\theta$.

$$\begin{aligned}
\int \tan^4\theta \sec^4\theta\, d\theta &= \int \tan^4\theta \sec^2(\theta)(\tan^2\theta + 1)\, d\theta \\
&= \int \tan^6\theta\sec^2\theta + \tan^4\theta\sec^2\theta\, d\theta \\
&= \int u^6 - u^4\, du = \frac{u^7}{7} + \frac{u^5}{5} \\
&= \frac{\tan^7\theta}{7} + \frac{\tan^5\theta}{5} + C.
\end{aligned}$$

Example 2.16. If we have an odd number of tangents we use our identity to rewrite things so we have exactly one tangent. We'll set $u = \sec(\theta)$ and $du = \sec(\theta)\tan(\theta)\, d\theta$.

$$\begin{aligned}
\int \tan^3\theta\sec^5\theta\, d\theta &= \int \tan\theta\sec^5\theta(\sec^2\theta - 1)\, d\theta \\
&= \int \tan\theta\sec^7\theta - \tan\theta\sec^5\theta\, d\theta \\
&= \int u^6 - u^4\, du = \frac{u^7}{7} - \frac{u^5}{5} \\
&= \frac{\sec^7\theta}{7} - \frac{\sec^5\theta}{5} + C.
\end{aligned}$$

Remark 2.17. If we have an even number of tangents and an odd number of secants, our life is hard; we usually use integration by parts. For instance,

$$\int \tan^2\theta\sec\theta\, d\theta = \frac{1}{2}\left(\tan\theta\sec\theta + \ln\left(\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right) - \ln\left(\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right)\right).$$

There are a couple of special cases, as well. We observed earlier (by setting $u = \cos\theta, du = -\sin\theta$) that

$$\int \tan\theta\, d\theta = -\ln|\cos\theta| = \ln|\sec\theta| + C.$$

More difficult is the integral of $\sec\theta$. This was a major open problem for most of a century, since it's important in the design of maps and in navigation. It was solved by James Gregroy in 1668, but Isaac Barrow came up with a clearer argument in 1670, by inventing the technique of *partial fractions* which we'll discuss in section 2.3.

For right now I'll just give you the formula:

$$\begin{aligned}
\int \sec\theta\, d\theta &= \ln|\sec\theta + \tan\theta| + C && \text{Gregory's formula} \\
&= \ln\left|\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\right| + C && \text{Numeric conjecture by Henry Bond} \\
&= \frac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| + C && \text{Barrow's formula.}
\end{aligned}$$

All three formulas here are valid. And all three formulas are obnoxious and seem really random.

2.2.2 Trigonometric Substitution

Now that we know how to integrate trigonometric functions, we can often use them to make our lives easier in integrals that don't appear to use trigonometry at all.

Example 2.18. We've known since grade school that the area of a circle with radius r is πr^2. Can we prove this? Consider the function $f(x) = \sqrt{r^2 - x^2}$; this is the graph of a semicircle over the x axis. So we wish to compute $\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$.

There's no way to use integration by parts. We might try setting $u = r^2 - x^2$ but then $du = -2x\, dx$ and we have no way to get rid of the x.

Instead, we do something clever. We notice that $\sqrt{r^2 - x^2}$ looks kind of like $\sqrt{r^2 - (r\sin(x))^2}$, and by the pythagorean identity that would simplify to $\sqrt{r\cos^2(x)} = \cos(x)$. Thus we write $x = r\sin\theta$, and $dx = r\cos\theta\, d\theta$. We get

$$\begin{aligned}
\int \sqrt{r^2 - x^2}\, dx &= \int \sqrt{r^2(1 - \sin^2\theta)} \cdot r\cos\theta\, d\theta \\
&= \int r^2\sqrt{\cos^2\theta} \cdot \cos\theta\, d\theta \\
&= r^2 \int \cos^2\theta\, d\theta \\
&= r^2 \int \frac{1 + \cos(2\theta)}{2}\, d\theta \\
&= r^2\left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right).
\end{aligned}$$

At this point we have two choices. One is to change the θ back into x by the formula $\theta = \arcsin(x/r)$. If we do this we find the antiderivative is

$$r^2\left(\frac{1}{2}\arcsin\left(\frac{x}{r}\right) + \frac{1}{4}\sin\left(2\arcsin\left(\frac{x}{r}\right)\right)\right).$$

In principle we can use the double-angle formulas to calculate $\sin(2\arcsin(x/r))$, but in practice this is a huge pain.

But when we do a u-substitution for a definite integral, we can *either* subtitute x back in for u, *or* change the bounds of the definite integral to be in terms of u. The second approach is much easier here. Our original integral was from $-r$ to r; we see that if $x = -r$ then $\theta = -\pi/2$, and if $x = r$ then $\theta = \pi/2$. So we evaluate this integral at $\pi/2$ and $-\pi/2$, and we get

$$\begin{aligned}\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx &= r^2 \left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4} \right) \Bigg|_{-\pi/2}^{\pi/2} \\ &= r^2 \left(\frac{\pi}{4} + \frac{\sin(\pi)}{4} - \left(\frac{-\pi}{4} + \frac{\sin(-\pi)}{4} \right) \right) = r^2 \cdot \frac{\pi}{2}.\end{aligned}$$

Thus the area of the semicircle is $\pi r^2/2$, and so the area of the circle is πr^2.

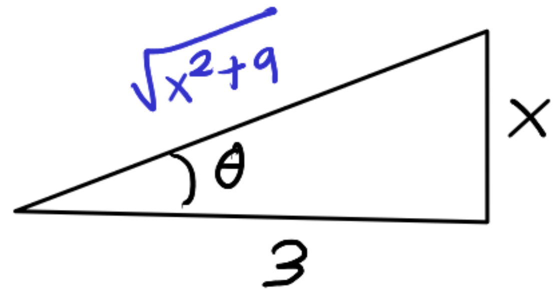
Remark 2.19. In general, this helps when we have a difference of squares under a square root.

- If we have $\int \sqrt{a^2 - x^2}\, dx$ we use $x = a \sin\theta$ (as above).
- If we have $\int \sqrt{a^2 + x^2}\, dx$ we use $x = a \tan\theta$.
- If we have $\int \sqrt{x^2 - a^2}\, dx$ we use $x = a \sec\theta$.

Example 2.20. Suppose we have $\displaystyle\int \frac{1}{x^2\sqrt{9 + x^2}}\, dx$. Then we set $x = 3\tan\theta, dx = 3\sec^2\theta\, d\theta$ and have

$$\begin{aligned}\int \frac{1}{x^2\sqrt{9+x^2}}\, dx &= \int \frac{1}{9\tan^2\theta\sqrt{9(1+\tan^2\theta)}} \cdot 3\sec^2\theta\, d\theta \\ &= \int \frac{\sec^2\theta}{3\tan^2\theta\sqrt{9\sec^2\theta}}\, d\theta \\ &= \int \frac{\sec\theta}{9\tan^2\theta}\, d\theta = \int \frac{\cos\theta}{9\sin^2\theta}\, d\theta \\ &= \int \frac{1}{9u^2}\, du \qquad \text{where } u = \sin\theta,\ du = \cos\theta\, d\theta \\ &= -\frac{1}{9u} + C = \frac{-1}{9\sin\theta} + C = -(\csc\theta)/9 + C.\end{aligned}$$

Now we just need to figure out what $\csc\theta$ is. But we know $\tan\theta = x/3$, so we can draw a right triangle with an angle θ, where the opposite side has length x and the adjacent side has length 3, and thus the hypotenuse has length $\sqrt{x^2+9}$.



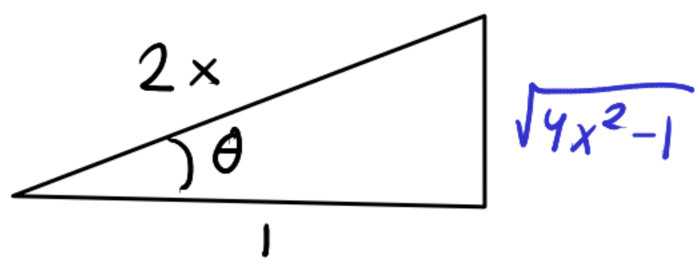
Then $\csc\theta = \sqrt{x^2+9}/x$ and thus we have

$$\int \frac{1}{x^2\sqrt{9+x^2}}\,dx = \frac{-\sqrt{x^2+9}}{9x} + C.$$

Example 2.21. Suppose we have $\int \frac{dx}{\sqrt{4x^2-1}}$. Then we can take $x = \frac{1}{2}\sec\theta$ and $dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$. We have

$$\begin{aligned}
\int \frac{dx}{\sqrt{4x^2-1}} &= \int \frac{\frac{1}{2}\sec\theta\tan\theta\,d\theta}{\sqrt{\sec^2\theta-1}} \\
&= \frac{1}{2}\int \frac{\sec\theta\tan\theta\,d\theta}{\sqrt{\tan^2\theta}} \\
&= \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta+\tan\theta| + C.
\end{aligned}$$

We know that $\sec\theta = 2x$ by our definition of θ. To find $\tan\theta$ we draw a triangle: angle θ has hypotenuse $2x$ and adjacent side 1, and thus opposite side $\sqrt{4x^2-1}$, so $\tan\theta = \sqrt{4x^2-1}$. Thus



$$\int \frac{dx}{\sqrt{4x^2-1}}\,dx = \ln|x+\sqrt{4x^2-1}/2| + C.$$

2.3 Integration by Partial Fractions

A last major technique deals with irritating fractions. It's relatively straightforward to compute

$$\int \frac{1}{x-1}\,dx = \ln|x-1| + C$$
$$\int \frac{x}{x^2-1}\,dx = \frac{1}{2}\ln|x^2-1| + C,$$

but it's considerably less clear how to integrate $\frac{2}{x^2-1}$. But if we somehow notice that $\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}$, then we have

$$\int \frac{2}{x^2-1}\,dx = \int \frac{1}{x-1} - \frac{1}{x+1}\,dx = \ln(x-1) - \ln(x+1) + C.$$

We want to find a way to break any fraction we have to deal with into simple fractions like those.

2.3.1 Polynomial Long Division

As a warmup, we need to remember (or learn for the first time) polynomial long division. Suppose we have a ratio of polynomials, and the numerator is higher degree than the denominator. We can split the ratio into a polynomial, plus a ratio where the numerator is smaller degree than the denominator.

Example 2.22. Consider $\dfrac{x^3+2x^2+1}{x+1}$. Looking at this term by term, we get

$$\begin{aligned}\frac{x^3+2x^2+1}{x+1} &= x^2 + \frac{x^2+1}{x+1} \\ &= x^2 + x + \frac{-x+1}{x+1} \\ &= x^2 + x - 1 + \frac{2}{x+1}\end{aligned}$$

and thus

$$\int \frac{x^3+2x^2+1}{x+1}\,dx = \int x^2 + x - 1 + \frac{2}{x+1}\,dx = \frac{x^3}{3} + \frac{x^2}{2} - x + 2\ln(x+1) + C.$$

Example 2.23. Suppose we want to compute $\int \frac{x^3+1}{x^2+1}\,dx$. We need to do a long division here:

$$\begin{aligned}x^3+1 &= x(x^2+1) - x + 1 \\ \frac{x^3+1}{x^2+1} &= x - \frac{x}{x^2+1} + \frac{1}{x^2+1}\end{aligned}$$

and thus

$$\begin{aligned}\int \frac{x^3+1}{x^2+1} &= \int x - \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx \\ &= \frac{x^2}{2} - \frac{1}{2}\ln|x^2+1| + \arctan(x) + C.\end{aligned}$$

2.3.2 Partial Fraction Decomposition

Once we have a fraction where the numerator is lower degree than the denominator, we factor the denominator and pull the fraction apart. By the Fundamental Theorem of Algebra, we can always factor any polynomial into a product of linear and quadratic factors—that is, degree-1 and degree-2 polynomials.

If we are asked to integrate a rational function $\frac{P(x)}{Q(x)}$, we begin by factoring Q completely into a product of linear and quadratic polynomials. We try to write $\frac{P(x)}{Q(x)}$ as a sum of fractions whose denominators are distinct factors of Q.

Example 2.24. Suppose we wish to find $\int \frac{3x^2-1}{x^3-x}\,dx$. We note that the numerator is smaller in degree than the denominator, so we don't have to do long division. We see that the denominator factors into $x(x+1)(x-1)$. So we wish to solve the equation

$$\begin{aligned} \frac{3x^2-1}{x^3-x} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} \\ 3x^2-1 &= A(x^2-1) + B(x^2-x) + C(x^2+x) \end{aligned}$$

From this point, there are two different approaches you can take. One is to group like terms together and then get a system of equations to solve. So we get

$$3x^2-1 = (A+B+C)x^2 + (C-B)x - A.$$

For two polynomials to be equal, each of their coefficients need to be equal; so we get a system of equations

$$\begin{aligned} 3 &= A+B+C \\ 0 &= C-B \\ -1 &= -A. \end{aligned}$$

The third equation tells us that $A = 1$, and the second equation tells us that $C = B$; from that the first equation tells us that $2 = B + C = 2B$ and thus $B = C = 1$. So we can write

$$\begin{aligned} \int \frac{3x^2-1}{x^3-x}\,dx &= \int \frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-1}\,dx \\ &= \ln|x| + \ln|x+1| + \ln|x-1| + C. \end{aligned}$$

If you take Linear Algebra (Math 2184 or 2185) you will learn a systematic approach to solving systems of equations like this.

However, there's a usually-simpler way of approaching this. If we look back at our first equation

$$3x^2 - 1 = A(x^2 - 1) + B(x^2 - x) + C(x^2 + x)$$

we can try plugging in numbers for x. For instance, if we set $x = 0$ we get

$$-1 = A(0 - 1) + B(0 - 0) + C(0 + 0) = -A$$

and thus $A = 1$. Similarly we can plug in $x = 1$ to get $2 = 2C$, or $x = -1$ to get $2 = 2B$.

How did we pick these values for x? These are precisely the roots of $x(x+1)(x-1)$; they're the places where we'd be dividing by zero in the original denominator. So one more way to think about this is that we take our original equation and multiply through by the denominator of just one term:

$$\frac{3x^2 - 1}{x^3 - x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$
$$\frac{3x^2 - 1}{x^2 - 1} = A + \frac{Bx}{x+1} + \frac{Cx}{x-1}.$$

Now we see that plugging in $x = 0$ will no longer cause division-by-zero errors, but it will kill off everything on the right-hand side that didn't originally have an x term in the denominator.

If we have repeated factors in the denominator, things are a bit trickier. We need to have one fraction for each possible power of each linear factor. For instance, if we wish to integrate $\frac{1}{x^3(x-1)^3}$ we could write

$$\frac{1}{x^3(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2} + \frac{F}{(x-1)^3}.$$

Looking at this formula, we may decide that we do not, in fact, wish to integrate $\frac{1}{x^3(x-1)^3}$, and that this is why we have purchased a computer.

Example 2.25. If we have $\int \frac{2x+1}{x^3+2x^2+x}\,dx$, we see the bottom factors into $x(x+1)^2$, with roots $-1, 0$. So we write

$$\frac{2x+1}{x^3 + 2x^2 + x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$
$$2x + 1 = A(x^2 + 2x + 1) + B(x^2 + x) + C(x).$$

If we plug in $x = 0$ we get $1 = A$. If we plug in -1 we get $-1 = -C$ so $C = 1$.

But now we've run out of roots; how do we figure out what B is supposed to be? Well, we now know what A and C are, so we have

$$2x + 1 = x^2 + 3x + 1 + B(x^2 + x).$$

This still has to be true if we plug in any value for x, we can just pick our favorite value. I'll pick $x = 1$ again, and we get $3 = 5 + 2B$ and thus $B = -1$. So we have

$$\begin{aligned}\int \frac{2x+1}{x^3 + 2x^2 + x}\,dx &= \int \frac{1}{x} - \frac{1}{x+1} + \frac{1}{(x+1)^2}\,dx \\ &= \ln|x| - \ln|x+1| - \frac{1}{x+1} + C.\end{aligned}$$

Sometimes we're stuck with quadratic factors. We treat them the same way we did the linear factors, except now our numerators will have terms like $Ax+B$ instead of just solitary numbers.

Example 2.26. If we wish to find $\int \frac{3x-1}{x(x^2+1)}\,dx$, we write

$$\begin{aligned}\frac{3x-1}{x(x^2+1)} &= \frac{A}{x} + \frac{Bx+C}{x^2+1} \\ 3x - 1 &= A(x^2+1) + (Bx+C)x\end{aligned}$$

Here again, setting $x = 0$ gives that $-1 = A$; since we're out of roots, we maybe pick some other numbers to plug in. If $x = 1$ we get $2 = 2A + B + C$, and $A = -1$, so $B + C = 4$. If $x = 2$ then $5 = 5A + 4B + 2C$ so $2B + C = 5$. Combining these two equations gives us that $B = 1$ and thus $C = 3$. Thus

$$\int \frac{3x-1}{x(x^2+1)}\,dx = \int \frac{-1}{x} + \frac{x+3}{x^2+1}\,dx.$$

That last term looks hard to deal with, but becomes easier if we split the numerator up into two pieces. One becomes a derivative of arctan, and the other becomes a u-substitution.

$$\begin{aligned}\int \frac{3x-1}{x(x^2+1)}\,dx &= \int \frac{-1}{x} + \frac{x}{x^2+1} + \frac{3}{x^2+1}\,dx \\ &= -\ln|x| + \frac{1}{2}\ln|x^2+1| + 3\arctan(x) + C.\end{aligned}$$

Remark 2.27. Arguably, we weren't out of roots for x^3+x there; if we allow complex numbers we could take $i = \sqrt{-1}$ as a root. Plugging that in to the equation would give

$$\begin{aligned}3i - 1 &= A(-1+1) + (Bi + C)i \\ 3i - 1 &= -B + Ci\end{aligned}$$

and thus $C = 3$ and $B = 1$. In principle this is the more "correct" way to do this problem, but I'm not going to expect you to work with complex numbers.

Example 2.28. Consider $\int \frac{2x^2+10x+13}{x(x^2+6x+13)}\,dx$. We write

$$\begin{aligned}\frac{2x^2+10x+13}{x(x^2+6x+13)} &= \frac{A}{x} + \frac{Bx+C}{x^2+6x+13}\\ 2x^2+10x+13 &= A(x^2+6x+13) + (Bx+C)x.\end{aligned}$$

Plugging in $x = 0$ gives $A = 1$, leaving us with

$$x^2 + 4x = Bx^2 + Cx$$

and so $C = 4, B = 1$. So we have

$$\int \frac{2x^2+10x+13}{x(x^2+6x+13)}\,dx = \int \frac{1}{x} + \frac{x+4}{x^2+6x+13}\,dx.$$

The first bit is easy, but the second bit is tricky; we need to find a u we can substitute in.

Our life is much easier if the denominator is a sum of squares, so we try to write it that way by completing the square. We notice that $x^2 + 6x + 9 = (x+3)^2$, so the denominator is $(x+3)^2 + 4$; we try $u = x + 3, du = dx$. Then

$$\begin{aligned}\int \frac{x+4}{x^2+6x+13}\,dx &= \int \frac{u+1}{u^2+4}\,du\\ &= \int \frac{u}{u^2+4}\,du + \int \frac{1}{u^2+4}\,du\\ &= \frac{1}{2}\ln(u^2+4) + \frac{1}{2}\arctan(u/2) + C\\ &= \frac{1}{2}\left(\ln(x^2+6x+13) + \arctan\left(\frac{x+3}{2}\right)\right) + C\\ \int \frac{2x^2+10x+13}{x(x^2+6x+13)}\,dx &= \ln|x| + \frac{1}{2}\left(\ln(x^2+6x+13) + \arctan\left(\frac{x+3}{2}\right)\right) + C.\end{aligned}$$

Example 2.29. Consider $\int \frac{x^4+x^3+4x^2+x+1}{x(x^2+1)^2}\,dx$. We compute

$$\begin{aligned}\frac{x^4+x^3+4x^2+x+1}{x(x^2+1)^2} &= \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}\\ x^4+x^3+4x2+x+1 &= A(x^2+1)^2 + (Bx+C)(x^3+x) + (Dx+E)x\\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A\end{aligned}$$

So we see quickly that $A = 1$ and thus $B = 0$. Similarly, $C = 1$. This tells us that $E = 0$ and $D = 2$. Then we have

$$\int \frac{x^4 + x^3 + 4x^2 + 2x + 1}{x(x^2+1)}\,dx = \int \frac{1}{x} + \frac{1}{x^2+1} + \frac{2x}{(x^2+1)^2}\,dx$$
$$= \ln|x| + \arctan(x) - (x^2+1)^{-1} + C.$$

And finally, sometimes we have to combine all this with polynomial long division.

Example 2.30. Consider $\int \frac{x^3 + x^2 + 3x + 1}{x^2 + x}\,dx$.

We see that the numerator has higher degree than the denominator, so we should start by doing a polynomial long division. We work out that

$$x^3 + x^2 + 3x + 1 = (x^2 + x)(x) + 3x + 1$$
$$\frac{x^3 + x^2 + 3x + 1}{x^2 + x} = x + \frac{3x+1}{x^2+x}.$$

Now we can do a partial fractions decomposition

$$\frac{3x+1}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}$$
$$3x + 1 = A(x+1) + Bx.$$

Plugging in 0 gives $A = 1$, and plugging in -1 gives $-2 = -B$ or $B = 2$. Thus we have

$$\int \frac{x^3 + x^2 + 3x + 1}{x^2 + x}\,dx = \int x + \frac{1}{x} + \frac{2}{x+1}\,dx$$
$$= \frac{x^2}{2} + \ln|x| + 2\ln|x+1| + C.$$

A Brief Note on How to Cheat

We've developed a lot of techniques for evaluating integrals over the past couple of weeks. However, as good mathematicians we're also fundamentally lazy and would prefer to avoid work when we can manage it. There are two common solutions here.

First, your textbook has an extensive integral table, and even more extensive tables can be found online. It often requires some massaging to get your integral into the form of the table, but for complex integrals the table will be much easier than figuring things out from scratch. (For instance, the table incorporates the results of trig subsitution without making you work through it explicitly).

Second, computers are very good at doing integrals. Wolfram Alpha can often integrate a function for you, as can other computer tools. It's dangerous to become overly reliant on these tools—it's easy to make a mistake if you don't understand what's going on, and sometimes the computer will return the answer in a less useful form. They are very good for automated computations and checking your work, however.

A final cautionary note: there are some functions that don't have a nice closed-form antiderivative. Famously, there's no way to write $\int e^{x^2}\,dx$ in terms of "elementary functions." That doesn't mean there is no antiderivative; the obvious one is $\int_0^x e^{t^2}\,dt$. But while correct, that answer isn't terribly enlightening.

However, some of these non-antidifferentiable functions are very important. For instance, $f(x) = e^{-x^2}$ is the "bell curve" or "normal distribution" that is critical to statistics. We cannot antidifferentiate this, but we can still do some useful computations. Later on in section 5 we'll see how we can use "infinite series" to get a handle on this sort of function.

But right now we can take another approach: we can try to get an approximate answer to questions that we cannot solve exactly.

2.4 Numeric Integration

Sometimes we have a function which we, for some reason, can't compute an exact antiderivative of: either it is too difficult, or none exists, or we simply don't have enough data because we are using experimental measurements. In these cases we can use numerical methods to approximate the integral of a function.

In Calculus I we used the basic Riemann sum, generally defaulting to using the right endpoint as the sample point:

$$\int_a^b f(t)\,dt \approx R_n = \sum_{i=1}^n \Delta x \cdot f(x_i) \qquad \Delta x = \frac{b-a}{n}.$$

This is generally a pretty good approximation, but it can fail very badly if the right endpoints aren't representative samples of the function output. We can improve this easily, if slightly, by sampling at better locations: we lose slightly less information if we sample at the **midpoint** of each interval. If we're taking a limit, this doesn't matter, which is why we didn't worry about it in Calc 1. But when we're trying to get numeric estimates out of the Riemann sums, it does matter.

Definition 2.31. We can define the *midpoint approximation with n rectangles* of an integral to be

$$\int_a^b f(t)\,dt \approx M_n = \sum_{i=1}^n \Delta x \cdot f\left(\frac{x_i + x_{i-1}}{2}\right).$$

Example 2.32. Let's approximate $\int_0^4 4x^3\,dx$ using four intervals. With right endpoints we have

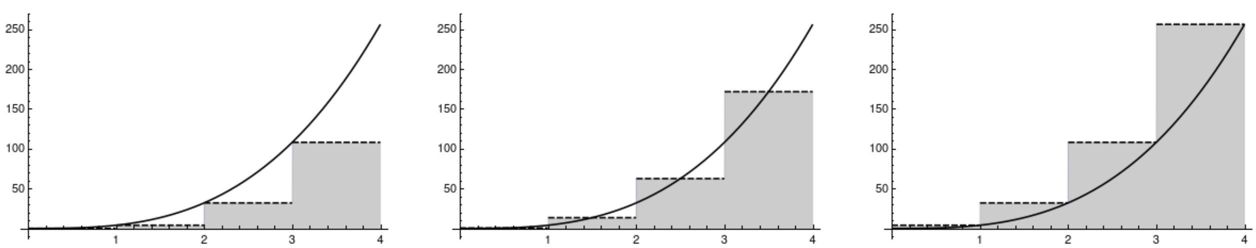
$$R_4 = \frac{4}{4}\left(f(1) + f(2) + f(3) + f(4)\right) = 4 + 32 + 108 + 256 = 400.$$

If we use midpoints instead we get

$$\begin{aligned} M_4 &= \frac{4}{4}\left(f\left(\frac{0+1}{2}\right) + f\left(\frac{1+2}{2}\right) + f\left(\frac{2+3}{2}\right) + f\left(\frac{3+4}{2}\right)\right) \\ &= .5 + 13.5 + 62.5 + 171.5 = 248. \end{aligned}$$

(The true answer, of course, is $x^4|_0^4 = 4^4 = 256$.)

Figure 2.1: The left endpoint, midpoint, and right endpoint Riemann sums for $\int_0^4 4x^3\,dx$. Notice how the midpoint sum visually has much less error than the other two.



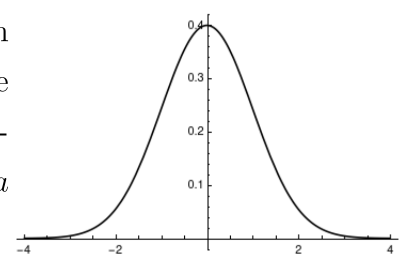
We might want to know how much error we can still have in this approximation. In general, if the function is jumping up and down wildly, we could have a ton of error; but if the function is tamer, we can keep the error fairly small.

Specifically, the extent to which the function can jump around is limited by the second derivative. If we have a number K such that $|f''(x)| \leq K$ on the interval of integration, then the error must be less than or equal to $\dfrac{K(b-a)^3}{24n^2}$.

In example 2.32, the second derivative is $24x$, and as long as $0 \leq x \leq 4$ we have $24x \leq 96$, so we take $K = 96$. Then the rule tells us that our error will be less than $\dfrac{96 \cdot 4^3}{12 \cdot 4^2} = 16$. Since our actual error is 8, this is true.

Example 2.33.

The *standard normal distribution* is defined by the function $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. The graph of this function is the bell curve you've probably seen if you've ever done any statistics, with standard deviation 1. The probability of getting a result between a and b standard deviations away from 0 is $\int_a^b \phi(t)\,dt$.



This is something we often need while doing statistics. For instance, we may want to know what the odds of being within one standard deviation of the mean is; this number is equal to $\int_{-1}^{1} \phi(t)\,dt$. If we had an antiderivative $\Phi(x)$ then this would be easy, but we can't actually antidifferentiate $e^{-x^2/2}$. (This isn't just a lack of tools on our part! It's a theorem that we can't write down an antiderivative using reasonable symbols and formulas.)

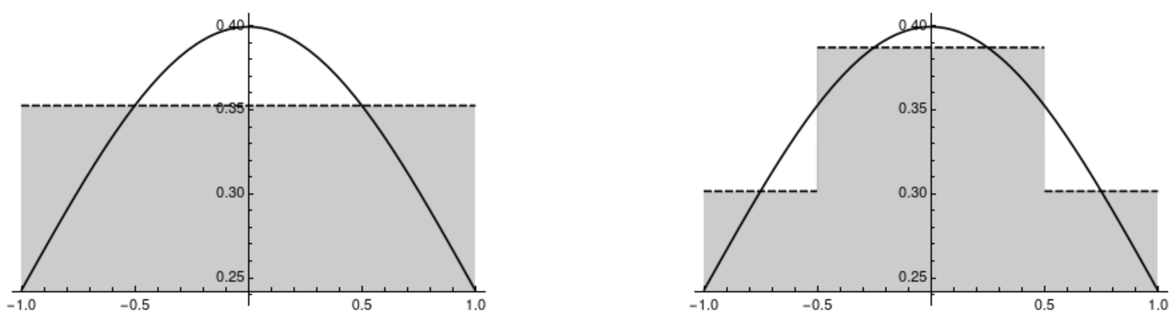
But this number is important, so we want an estimate of it anyway. We can use something like the midpoint rule to estimate it.

$$\int_{-1}^{1} \phi(t)\,dt \approx \phi(-.5)\cdot 1 + \phi(.5)\cdot 1 \approx .35 + .35 = .7$$

which should have relatively small error. But if we want it more accurate, we could do a midpoint approximation with more intervals:

$$\begin{aligned}\int_{-1}^{1} \phi(t)\,dt &\approx \phi(-.75)\cdot .5 + \phi(-.25)\cdot .5 + \phi(.25)\cdot .5 + \phi(.75)\cdot .5\\ &\approx .30\cdot .5 + .39\cdot .5 + .39\cdot .5 + .30\cdot .5 = .69.\end{aligned}$$

Figure 2.2: Midpoint approximation to the normal distribution, with two and four intervals



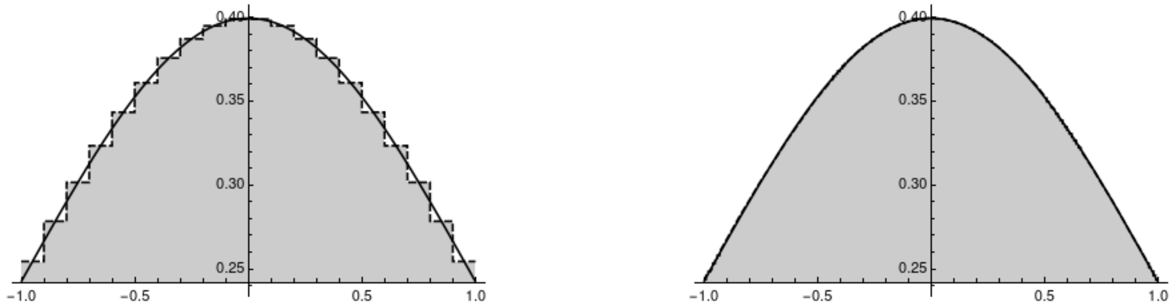
If we want a more precise estimate, we can always use more intervals. With twenty intervals, we get roughly .68—which is also what we get with 2000 intervals. Thus we can conclude there's a roughly 68% chance of landing within one standard deviation of the mean.

We can improve our approximations even more by moving away from rectangles altogether. Rather than sampling at one point, why not sample at *both* endpoints of the rectangle and average them? This leads to what is known as the **trapezoidal rule**:

$$\int_a^b f(t)\,dt \approx T_n = \sum_{i=1}^{n} \Delta x \cdot \frac{(f(x_{i-1}) + f(x_i))}{2}.$$

If you draw a picture it quickly becomes clear why this is a "trapezoidal" rule; we are taking the area of a trapezoid with base running from x_{i-1} to x_i and with top endpoints $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$.

Figure 2.3: Midpoint approximation to the normal distribution, with 20 and 200 intervals



This approximation has error $|E_T| \leq \frac{K(b-a)^3}{12n^2}$ if $|f''(x)| \leq K$. This is a worse error bound than the midpoint rule, but you can usually actually use more intervals with the trapezoid rule in practice.

Example 2.34. Let's approximate $\int_0^4 4x^3$ using four intervals, again. We get

$$\begin{aligned} T_4 &= \frac{4}{4}\left(\frac{f(0)+f(1)}{2} + \frac{f(1)+f(2)}{2} + \frac{f(2)+f(3)}{2} + \frac{f(3)+f(4)}{2}\right) \\ &= \frac{0+4}{2} + \frac{4+32}{2} + \frac{32+108}{2} + \frac{108+256}{2} = 272. \end{aligned}$$

Recall the true answer is 256, so we have error 16. This is no larger than the bound

$$E_T \leq \frac{96 \cdot 4^3}{12 \cdot 4^2} = 32.$$

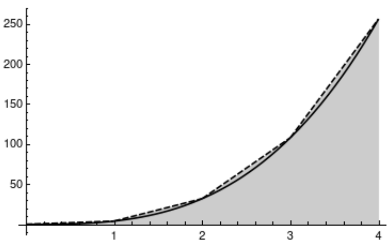


Figure 2.4: $\int_0^4 4x^3\,dx$ approximated with the trapezoid rule

Remark 2.35. There's a real sense in which the trapezoid rule isn't as good as the midpoint rule: we have $|E_M| \leq \frac{K(b-a)^3}{24n^2}$ while $|E_T| \leq \frac{K(b-a)^3}{12n^2}$, which is twice as big.

However, in a real sense, the midpoint rule requires twice as much *data* to work with—we need data on the midpoints, and not the endpoints, which generally means twice as many measurements. So the trapezoid rule can give better results in practice when working with real data.

We often want to use these techniques in real life when we're working from experimental data. Sometimes we have a bunch of specific measurements of the derivative, but we don't have an actual formula. Then we can't use the fundamental theorem of calculus to evaluate the integral exactly; but we can still approximate it from our data.

Example 2.36.

Suppose we have the speed of a runner at the following times:

$0s$	0	$.5s$	4.67	$1s$	7.34	$1.5s$	8.86	$2s$	9.73	$2.5s$	10.22
$3.0s$	10.51	$3.5s$	10.67	$4.0s$	10.76	$4.5s$	10.81	$5.0s$	10.81		

Can we estimate the distance covered?

This is in fact an integral: we're giving data about the velocity, or derivative, and we want to know about the distance, which is the original function. So we want to compute $\int_0^5 v(t)\,dt$. We can't possibly do a "real" integral because we don't have a formula for the whole function, but we can use the data we collected to estimate the integral.

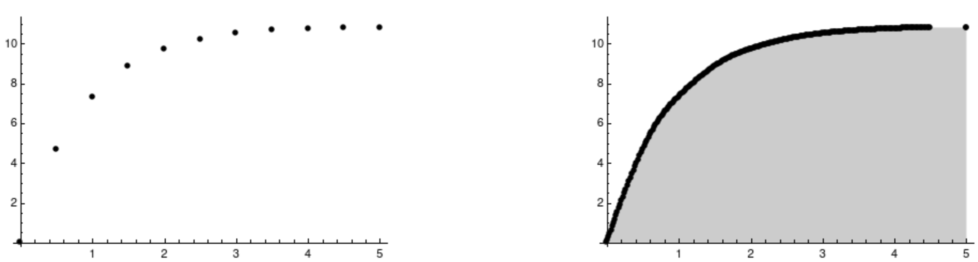


Figure 2.5: Left: Measurements of the runner's speed. Right: the shaded region is the distance covered

We can estimate using the Trapezoid rule:

$$\begin{aligned}
T_{10} &= \frac{1}{2}\left(\frac{0+4.67}{2} + \frac{4.67+7.34}{2} + \frac{7.34+8.86}{2} + \frac{8.86+9.73}{2} + \frac{9.73+10.22}{2} + \frac{10.22+10.51}{2}\right. \\
&\qquad \left. + \frac{10.51+10.67}{2} + \frac{10.67+10.76}{2} + \frac{10.76+10.81}{2} + \frac{10.81+10.81}{2}\right) \\
&= \frac{1}{2}\left(2.335 + 6.005 + 8.1 + 9.295 + 9.975 + 10.365 + 10.59 + 10.715 + 10.785 + 10.81\right) \\
&= 44.4875.
\end{aligned}$$

In contrast, if we want to use the midpoint approximation, we can only use five intervals.

$$M_5 = 1 \cdot \big(4.67 + 8.86 + 10.22 + 10.67 + 10.81\big) = 45.21$$

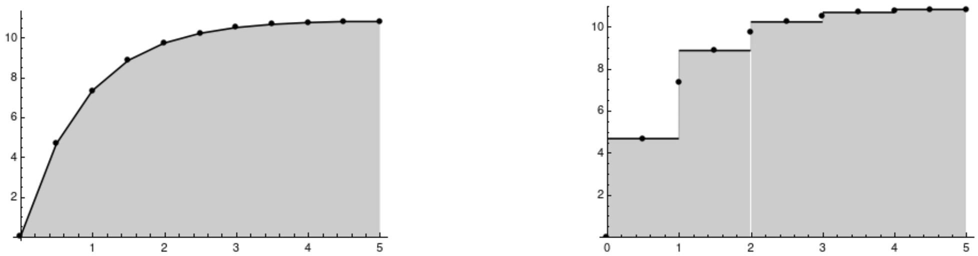


Figure 2.6: Left: an approximation with the trapezoid rule and ten intervals. Right: an approximation with the midpoint rule and five intervals.

If averaging two points is good, then averaging three points must be better, right? Rather than sampling one point, or making a trapezoid out of each pair of points, we can draw a parabola through each set of three points. A bit of algebra gives **Simpson's Rule**:

$$\begin{aligned}\int_a^b f(x)\,dx \approx S_n &= \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right) \\ &= \frac{\Delta x}{3}\left(f(x_0) - f(x_n) + \sum_{i=1}^{n/2} 4f(x_{2i-1}) + 2f(x_{2i})\right).\end{aligned}$$

(Note that this assumes n is even).

If $|f^{(4)}(x)| \leq L$ for $a \leq x \leq b$, and E_S is the error in Simpson's rule, then

$$|E_S| \leq \frac{L(b-a)^5}{180n^4}.$$

Example 2.37. What happens if we estimate our runner's speed with Simpson's rule? We get

$$\begin{aligned}S_{10} &= \frac{1}{6}(0 + 4 \cdot 4.67 + 2 \cdot 7.34 + 4 \cdot 8.86 + 2 \cdot 9.73 + 4 \cdot 10.22 \\ &\quad +2 \cdot 10.51 + 4 \cdot 10.67 + 2 \cdot 10.76 + 4 \cdot 10.81 + 10.81) \\ &= \frac{268.41}{6} = 44.735.\end{aligned}$$

If averaging three points is better, then averaging four must be even better, right? Well, technically, yes, but it's almost never worth the effort.

Example 2.38. Suppose we want to compute $\int_0^2 e^{x^2}$. How many intervals do we need, with each method, to guarantee the error is less than 1 in a thousand–that is, to guarantee the answer is correct to three decimal places?

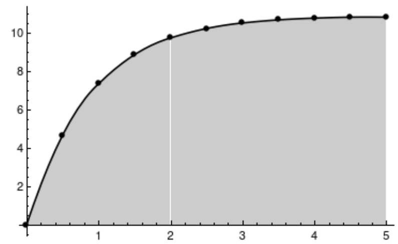


Figure 2.7: The runner's position estimated with Simpson's rule

On the interval, $f''(x) = 2e^{x^2} + 4x^2e^{x^2}$ is maximized when $x = 2$. $f''(2) = 18e^4 = K$. So we have

$$|E_M| \leq \frac{K \cdot 2^3}{24n^2} = \frac{6e^4}{n^2}$$
$$|E_T| \leq \frac{K \cdot 2^3}{12n^2} = \frac{12e^4}{n^2}.$$

Thus the trapezoid approximation is guaranteed to be accurate to within 1/1000 when $n > 809$, and the midpoint approximation is guaranteed to be accurate to within 1/100 when $n > 572$.

$f''''(x)$ is maximized at $x = 2$, where it is equal to $460e^4 = L$. Thus

$$|E_S| \leq \frac{L \cdot 2^5}{180n^4} = \frac{736e^4}{9n^4}.$$

Thus the Simpson's rule approximation is guaranteed to be accurate to within 1/1000 when $n > 45$.

So to get an answer to within 1/1000 we can use a computer to compute

$$S_{46} = \frac{2}{46 \cdot 3}\left(f(0) - f(2) + \sum_{i=1}^{23} 4f\left(\frac{2(2i-1)}{46}\right) + 2f\left(\frac{2 \cdot 2i}{46}\right)\right) = \frac{2 \cdot 1135.24}{46 \cdot 3} = 16.4528.$$

Note that since the true answer is 16.4526, this is accurate within $2/10,000$, which is what we wanted.