# Math 1232 Spring 2024
# Single-Variable Calculus 2 Section 12
# Mastery Quiz 10
# Due Tuesday, April 9

This week's mastery quiz has three topics. Everyone should submit work on all three.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

**Topics on This Quiz**

- Major Topic 3: Series Convergence
- Major Topic 4: Taylor Series
- Secondary Topic 8: Power Series

## Name:

## Recitation Section:

# M3: Series Convergence

(a) Analyze the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{5^n+1}$

**Solution:** We use the Ratio test. We have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(-1)^{n+1}3^{n+1}/5^{n+1}+1}{(-1)^n 3^n/5^n+1}\right| &= \lim_{n\to\infty}\frac{3^{n+1}(5^n+1)}{3^n(5^{n+1}+1)}\\ &= \lim_{n\to\infty} 3\frac{5^n+1}{5^{n+1}+1}\\ &= \lim_{n\to\infty} 3\frac{1+1/5^n}{5+1/5^n} = \frac{3}{5}.\end{aligned}$$

This limit is less than 1, so by the ratio test this converges absolutely.

(b) Analyze the convergence of the series $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

**Solution:** We can work this out with the integral test. We have

$$\begin{aligned}\int_1^{\infty} xe^{-x^2}\,dx &= \lim_{t\to\infty}\int_1^t xe^{-x^2}\,dx\\ &= \lim_{t\to\infty}\frac{-1}{2}e^{-x^2}\Big|_1^t\\ &= \lim_{t\to\infty}\frac{1}{2e}-\frac{1}{2e^{t^2}} = \frac{1}{2e} < \infty.\end{aligned}$$

Since this integral converges, the series must also converge by the integral test.

(c) Analyze the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n+1}$

**Solution:** You might try the ratio test here, but it won't actually help:

$$\lim_{n\to\infty}\left|\frac{(-2)^{n+1}/(n+1)2^{n+1}+1}{(-2)^n/n2^n+1}\right| = \lim_{n\to\infty}\frac{2(n2^n+1)}{(n+1)2^{n+1}+1} = \lim_{n\to\infty}\frac{n+1/2^{n+1}}{n+1+1/2^{n+1}} = 1.$$

Instead, we observe that this is an alternating series with the terms tending to zero, since

$$\lim_{n\to\infty}\frac{(-2)^n}{n2^n+1} = \lim_{n\to\infty}\frac{(-1)^n}{n+1/2^n} = 0.$$

Thus it converges. However, if we look at the absolute value, we can compare it to the series $\sum \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{2^n/n2^n+1}{1/n} = \lim_{n\to\infty} \frac{n2^n}{n2^n+1} = 1$$

and since $\sum \frac{1}{n}$ diverges, by the limit comparison test our absolute-value series also diverges. Thus the original series converges conditionally.

# M4: Taylor Series

(a) Write a power series expression for $\frac{2x^2}{4x+1}$ centered at 0. What is the radius of convergence?

**Solution:** We know that

$$\begin{aligned}
\frac{1}{1-(-4x)} &= \sum_{n=0}^{\infty}(-4x)^n \\
\frac{2x^2}{1+4x} &= 2x^2\sum_{n=0}^{\infty}(-4)^n x^n \\
&= \sum_{n=0}^{\infty} 2\cdot(-4)^n x^{n+2} \\
\text{(or)} \qquad &= \sum_{n=2}^{\infty} 2^{2n-3}(-1)^n x^n.
\end{aligned}$$

The radius of convergence is $1/4$. We can figure that out by reasoning from the geometric series: the radius of convergence for the geometric series is 1, so it converges for $-1 < -4x < 1$ or $-1/4 < x < 1/4$. Or we can use the ratio test:

$$\lim_{n\to\infty}\left|\frac{2^{2n-1}(-1)^{n+1}x^{n+1}}{2^{2n-3}(-1)^n x^n}\right| = \lim_{n\to\infty} 4|x|$$

and thus it converges when $4|x| < 1$.

(b) Write a power series expression for $\ln(1+x^2)$ centered at 0. What is the radius of convergence?

**Solution:** There are a few ways to approach this. One is to observe that $\frac{d}{dx}\ln(1+x^2) = \frac{2x}{1+x^2}$. We know that

$$\begin{aligned}
\frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} \\
\frac{2x}{1+x^2} &= 2x\sum_{n=0}(-1)^n x^{2n} = \sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}
\end{aligned}$$

Then we can integrate both sides:

$$\ln(1+x^2) = \int \sum_0^\infty 2(-1)^n (x)^{2n+1}\, dx$$
$$= C + \sum_0^\infty 2(-1)^n \frac{x^{2n+2}}{2n+2}$$

and plugging 0 in on both sides tells us that $C = 0$. So our power series is

$$\ln(1+x^2) = \sum_0^\infty 2(-1)^n \frac{x^{2n+2}}{2n+2}.$$

Alternatively, you could recall that

$$\ln(1+x) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} x^n$$

so we can compute

$$\ln(1+x^2) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} x^n$$
$$= \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} x^{2n}.$$

This looks different from our previous answer because it's been reindexed, but it is in fact the same answer. As an exercise, see if you can show they're the same!

(c) If $f(x) = \sum_{n=0}^\infty \frac{n+1}{n!+1} x^n$, compute $\int_3^6 f(x)$. (Hint: you can give your answer in the form of a series.)

**Solution:**

$$\int f(x) = \sum_{n=0}^\infty \frac{1}{n!+1} x^{n+1} + C$$
$$\int_3^6 f(x) = \sum_{n=0}^\infty \frac{1}{n!+1}(6^{n+1} - 3^{n+1}).$$

# S8: Power Series

(a) Find the radius of convergence and the interval of convergence of $\sum_{n=0}^\infty \frac{2^n}{n^2+n} x^n$.

**Solution:** We use the ratio test.

$$\lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}/(n+1)^2+n+1}{2^n x^n/n^2+n}\right| = \lim_{n\to\infty} 2|x|\frac{n^2+3n+2}{n^2+n}$$
$$= 2|x|.$$

So we need $2|x| < 1$ or $-1 < 2x < 1$, or $-1/2 < x < 1/2$ We need to have $x$ in the interval $(0-1/2, 0+1/2)$, so the radius is $1/2$.

To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty}\frac{2^n}{n^2+n}(1/2)^n \sum_{n=0}^{\infty}\frac{1}{n^2+n}$$

converges by comparison to a $p$-series

$$\sum_{n=0}^{\infty}\frac{2^n}{n^2+n}(-1/2)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+n}$$

converges by the alternating series test

Thus the interval of convergence is $[-1/2, 1/2]$.

(b) Find the radius of convergence and the interval of convergence of $\displaystyle\sum_{n=0}^{\infty}\frac{(n!)^2}{(3n)!}(x+2)^n$.

**Solution:** We use the ratio test.

$$\lim_{n\to\infty}\left|\frac{((n+1)!)^2(x+2)^{n+1}/(3n+3)!}{(n!)^2(x+2)^n/(3n)!}\right| = \lim_{n\to\infty}|x+2|\frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} \leq \frac{|x+2|}{n} = 0$$

for any $x$. So the radius of convergence is infinity, and this converges for all $x$.