Math 1232: Single-Variable Calculus 2
George Washington University Spring 2024
Recitation 10

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Problem 1. For each of the following series, write a careful argument showing either that it converges or that it diverges. Think about exactly what test you want to use and why.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{5n^3 - 2}{3n^5 - n}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{n^3 \ln(n) + 1}{n^4 - 7}$.

Solution:

(a) This is a pile of polynomials, so it'll be simplest to use the limit comparison test. It looks like $\frac{n^3}{n^5} = \frac{1}{n^2}$, so we compute

$$\lim_{n \to \infty} \frac{\frac{5n^3-2}{3n^5-n}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{5n^5 - 2n^3}{3n^5 - n} = 5/3.$$

This is a real finite non-zero limit. Then since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges as a p-series, our original series converges by the Limit Comparison Test.

(b) This doesn't look at all geometric, but also isn't just polynomials, so we hope the regular comparison test works. This looks kinda like $\frac{n^3}{n^4} = \frac{1}{n}$. And in fact we see $n^3 \ln(n) + 1 > n^3 \ln(n) > n^3$ as long as $n > 2$, and $n^4 - 7 < n^4$. So we have

$$\frac{n^3 \ln(n) + 1}{n^4 - 7} > \frac{n^3}{n^4} = \frac{1}{n}.$$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by the p-series test, we know that our original series diverges by the comparison test.

Problem 2. Analyze the convergence of the following series. Write clean arguments that establish whether they diverge, converge conditionally, or converge absolutely. Think about what tools/tests you want to use, and why.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{2^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n^2+e^n)}{n^2}$

(c) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{\ln(n)}$

Solution:

(a) Here we can start with the divergence test. We know that

$$n! = n(n-1)(n-2)\dots(3)(2)(1) \geq n(2)(2)\dots(2)(2)1 = n2^{n-2}$$

and thus $\frac{n!}{2^n} \geq \frac{n}{4}$. Thus

$$\lim_{n\to\infty} \frac{n!}{2^n} \geq \lim_{n\to\infty} \frac{n}{4} = +\infty \neq 0$$

and so this series diverges by the divergence test.

(b) This is a non-positive series, and it's not alternating, so we need to think about absolute convergence.

We have that $\left|\frac{\sin(n^2+e^n)}{n^2-n}\right| =\leq \frac{1}{n^2}$, so by the comparison test the series $\sum_{n=1}^{\infty}\left|\frac{\sin(n^2+e^n)}{n^2-n}\right|$ converges. Thus our series converges absolutely. Thus it converges.

(c) First we consider absolute convergence. We know that $\frac{1}{\ln(n)} \geq \frac{1}{n}$ when $n \geq 3$, and $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges. So by the comparison test $\sum_{n=3}^{\infty} \frac{1}{\ln(n)}$ diverges, and so our original series does not converge absolutely.

Now we consider conditional convergence. $\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$, and thus by the Alternating Series Test, our series converges.

Therefore our original series converges conditionally.

Problem 3. Let r be a real number. Does $\sum_{n=1}^{\infty} \frac{r^n}{n!}$ converge? What test do we want to use, and why? Does the answer depend on the value of r?

Solution: This has a geometric-y bit (the r^n), and a factorial bit, so it really wants the ratio test.

We have

$$\lim_{n\to\infty}\left|\frac{r^{n+1}/(n+1)!}{r^n/n!}\right| = \lim_{n\to\infty}\frac{|r|}{n+1} = 0 < 1.$$

Regardless of the value of r this limit is zero, so by the ratio test, this series converges absolutely for any r.

Problem 4. In class we showed that $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges to a number between 1 and 1/2. I claimed that it converged to $\ln 2$. But this convergence is conditional, and that matters.

(a) Write out the first twelve terms of this series.

(b) Reorganize them so that you have the same collection of numbers add one and then subtract two, then add one, then subtract two, and so on. (You'll have some extras left over and that's fine; remember you have an infinite list of terms.)

(c) What does each triplet look like? Can you simplify that somehow so it looks like something we recognize? (Hint: what happens if you combine the first two terms of a triplet?)

(d) Can you figure out what this sequence of partial sums converges to?

Solution:

(a)
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \dots$$

(b) We get
$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \dots$$
The 7, 9, and 11 terms are left over, but if we kept going we'd get $\frac{1}{7} - \frac{1}{14} - \frac{1}{16}$ and so on.

(c) We see that each triplet has the pattern $\frac{1}{n} - \frac{1}{2n} - \frac{1}{2n+2}$, where n is an odd number. If we collect the first two terms, we get $\frac{1}{2n} - \frac{1}{2n+2}$. So the first few triplets are
$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12}$$

(d) This is exactly half of our original series. So when we add them up in this order, we get $\frac{\ln(2)}{2}$.