Math 1232 Spring 2024
Single-Variable Calculus 2 Section 12
Mastery Quiz 11
Due Tuesday, April 16

This week's mastery quiz has three topics. Everyone should submit topic M4. If you have a 4/4 on M3, or a 2/2 on S8, you don't need to submit them.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Major Topic 4: Taylor Series
- Secondary Topic 8: Power Series

Name:

Recitation Section:

M3: Series Convergence

(a) Analyze the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$

Solution: This is an alternating series. Since the terms $\frac{n}{n^2+1}$ tend to zero as n goes to infinity, this converges by the alternating series test.

However, it doesn't absolutely converge. If we look at the absolute value series, we have $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. You can see this doesn't converge in a couple ways. The integral test would work. The regular comparison test will *not* work unless you're really careful: $\frac{n}{n^2+1} < \frac{1}{n}$ so we'd need to do some chicanery.

So it seems like this calls for the limit comparison test. We have

$$\lim_{n\to\infty} \frac{{}^{n}/{n^2+1}}{{}^{1}/{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1.$$

Since the harmonic series $\sum \frac{1}{n}$ diverges, by the limit comparison test, $\sum \frac{n}{n^2+1}$ diverges, and thus our series does not converge absolutely.

(b) Analyze the convergence of the series $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\ln(n)}$

Solution: By L'Hospital's rule, we compute that

$$\lim_{n\to\infty} \frac{n}{\ln(n)} = \lim_{x\to+\infty} \frac{x^{\nearrow\infty}}{\ln(x)_{\searrow\infty}} \stackrel{\text{L'H}}{=} \lim_{x\to+\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty.$$

Since this isn't zero, the series diverges by the divergence test.

(c) Analyze the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^4+1}$

Solution: We first consider the series of the absolute values, which is $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$. For this we can use the integral test:

$$\begin{aligned}\int_1^{\infty} \frac{x}{x^4+1}\,dx &= \lim_{t\to\infty} \int_1^t \frac{x}{x^4+1}\,dx \\ &= \lim_{t\to\infty} \frac{1}{2}\arctan(x^2)\Big|_1^t = \lim_{t\to\infty} \frac{1}{2}\left(\arctan(t^2) - \arctan(1)\right) \\ &= \frac{1}{2}\left(\pi/2 - \pi/4\right) = \frac{\pi}{8}.\end{aligned}$$

This is finite and convergent, so by the integral test, the series $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ converges.

Alternatively, we could observe that $\frac{n}{n^4+1} \le \frac{1}{n^3}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by the p-series test, we can conclude that $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ converges by the comparison test.

Thus our original series converges absolutely.

M4: Taylor Series

(a) In class we computed a Taylor series for $\sin(x)$ centered at zero. Use the degree-seven Taylor polynomial to approximate $\sin(3) \approx T_7(3, 0)$. (You don't need to numerically simplify this.)

Using the Taylor series remainder, find an upper bound for the error in this approximation.

Solution: We know that

$$\begin{aligned}
\sin(x) &= \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
T_7(x, 0) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\
T_7(x, 3) &= 3 - \frac{27}{3!} + \frac{3^5}{5!} - \frac{3^7}{7!} = 3 - \frac{37}{6} + \frac{243}{120} - \frac{2187}{5040} \\
&= 3 - \frac{9}{2} + \frac{81}{40} - \frac{243}{560} = \frac{51}{560} \approx 0.091.
\end{aligned}$$

We know that $f^{n+1}(x) = \pm\cos(x)$ or $\pm\sin(x)$ so $|f^{n+1}(z)| \leq 1$, and thus

$$\begin{aligned}
|R_n(x)| &= \left| \frac{f^{(n+1)(z)}}{(n+1)!} x^{n+1} \right| \leq \frac{x^{n+1}}{(n+1)!} \\
|R_7(x)| &\leq \frac{x^{7+1}}{(7+1)!} \\
|R_7(3)| &\leq \frac{3^8}{8!} = \frac{729}{4480} \approx 0.16.
\end{aligned}$$

It would also be okay to observe that the eighth term is zero, so we could actually compute

$$\begin{aligned}
|R_n(x)| &= \left| \frac{f^{(n+1)(z)}}{(n+1)!} x^{n+1} \right| \leq \frac{x^{n+1}}{(n+1)!} \\
|R_8(x)| &\leq \frac{x^{8+1}}{(8+1)!} \\
|R_8(3)| &\leq \frac{3^9}{9!} = \frac{243}{4480} \approx 0.054.
\end{aligned}$$

(b) Using series we already know, write down a formula for the (infinite) Taylor series for $(1+3x)^{2/3}$, and then write down the degree-three polynomial explicitly.

Solution: We can take this from the binomial series. So we have

$$f(x) = \sum_{n=0}^{\infty} \binom{2/3}{n} (3x)^n = \sum_{n=0}^{\infty} \binom{2/3}{n} (3)^n x^n$$

$$T_3(x,0) = 1 + \frac{2/3}{1} \cdot 3x + \frac{(2/3)(-1/3)}{2} \cdot 3^2 x^2 + \frac{(2/3)(-1/3)(-4/3)}{6} \cdot 3^3 x^3$$

$$= 1 + 2x - x^2 + \frac{4}{3} x^3.$$

(c) Using series we already know, write down a formula for the (infinite) Taylor series for $x^2 \ln(1-2x^3)$, and then write down the degree-eleven polynomial explicitly.

Solution: We can take this from the series for $\ln(1+x)$. So we have

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$\ln(1-2x^3) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-2x^3)^n}{n}$$

$$x^2 \ln(1-2x^3) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n 2^n x^{3n+2}}{n}$$

$$= \sum_{n=1}^{\infty} \frac{-2^n}{n} x^{3n+2} T_{11}(x,0) \qquad = -2x^5 - 2x^8 - \frac{8}{3} x^{11}.$$

S8: Power Series

(a) Find the radius of convergence and the interval of convergence of $\sum_{n=0}^{\infty} (n(x-3))^n$.

Solution: We use the ratio test.

$$\lim_{n\to\infty} \left| \frac{(n+1)^{n+1}(x-3)^{n+1}}{n^n (x-3)^n} \right| = \lim_{n\to\infty} |x-3| \left(\frac{n+1}{n} \right)^n (n+1) \quad \geq |x-3| \lim_{n\to\infty} (n+1) = \infty$$

unless $x = 3$. So the radius of convergence is 0, and the series converges if and only if $x = 3$.

(b) Find the radius of convergence and the interval of convergence of $\sum_{n=0}^{\infty} \frac{(5x-3)^n}{\sqrt{n}}$.

Solution: We use the ratio test.

$$\lim_{n\to\infty}\left|\frac{(5x-3)^{n+1}/\sqrt{n+1}}{(5x-3)^n/\sqrt{n}}\right| = \lim_{n\to\infty}\left|\frac{(5x-3)\sqrt{n}}{\sqrt{n+1}}\right|$$
$$= |5x-3|\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = |5x-3|.$$

So we need $|5x-3|<1$ or $-1<5x-3<1$, or $2<5x<4$ or $2/5<x<4/5$. We need to have x in the interval $(3/5-1/5, 3/5+1/5)$, so the radius is $1/5$.

To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty}\frac{(2-3)^n}{\sqrt{n}}\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n}}$$

converges by alternating series test

$$\sum_{n=0}^{\infty}\frac{(4-3)^n}{\sqrt{n}} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n}}$$

diverges by p-series test

Thus the interval of convergence is $[2/5, 4/5)$.