Math 1232 Spring 2024
Single-Variable Calculus 2 Section 12
Mastery Quiz 12
Due Tuesday, April 23

This week's mastery quiz has four topics. Everyone should submit topics S9 and S10.. If you have a 4/4 on M3 or M4, you don't need to submit them.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Major Topic 4: Taylor Series
- Secondary Topic 8: Power Series

Name:

Recitation Section:

M3: Series Convergence

(a) Analyze the convergence of the series $\sum_{n=1}^{\infty} \frac{n^2+n-3}{n^2 4^n}$

Solution: We use the ratio test. We have

$$L = \lim_{n\to\infty} \left| \frac{(n+1)^2+n+1-3/(n+1)^2 4^{n+1}}{n^2+n-3/n^2 4^n} \right|$$
$$= \lim_{n\to\infty} \left| \frac{(n^2+2n-1)n^2 4^n}{(n+1)^2 4^{n+1}(n^2+n-3)} \right|$$
$$= \lim_{n\to\infty} \left| \frac{n^4+2n^3-n^2}{4n^4+12n^3-20n-15} \right| = \frac{1}{4} < 1.$$

So by the ratio test this converges absolutely..

(b) Analyze the convergence of the series $\sum_{n=2}^{\infty} \frac{\ln(n)+n}{n^2-1}$

Solution: You can't really use the limit comparison test here, at least not easily, because the numerator is a bit over-complicated. But you can use the usual comparison test. We know that $n \leq n + \ln(n)$ and $n^2 - 1 < n^2$, so

$$\frac{\ln(n)+n}{n^2-1} \geq \frac{n}{n^2} = \frac{1}{n}.$$

We know that $\sum \frac{1}{n}$ diverges by the p-series test, so our series diverges by the comparison test.

(c) Analyze the convergence of the series $\sum_{n=1}^{\infty} (-1)^n \frac{3n^4-1}{n^5+1}$.

Solution: We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. But

$$\lim_{n\to\infty} \frac{3n^4-1/n^5+1}{1/n} = \lim_{n\to\infty} \frac{3n^5-n}{n^5+1} = 3$$

which is a finite non-zero number, so by the limit comparison test, $\sum_{n=1}^{\infty} \frac{3n^4-1}{n^5+1}$ diverges. So the series does not converge absolutely.

However, $\lim_{n\to\infty} \frac{3n^4-1}{n^5+1} = 0$, so by the alternating series test, $\sum_{n=1}^{\infty} (-1)^n \frac{3n^4-1}{n^5+1}$ converges. So the series converges conditionally.

M4: Taylor Series

(a) Let $f(x) = \cos^2(x)$. Use *the definition of a Taylor series* to find $T_4(x, \pi)$ for this function. (That is, find the terms up through the degree four term.)

Solution:

$$\begin{aligned} f(x) &= \cos^2(x) & f(\pi) &= 1 \\ f'(x) &= -2\cos(x)\sin(x) & f'(\pi) &= 0 \\ f''(x) &= 2\sin^2(x) - 2\cos^2(x) & f''(\pi) &= -2 \\ f'''(x) &= 4\sin(x)\cos(x) + 4\cos(x)\sin(x) & f'''(\pi) &= 0 \\ f^{(4)} &= 8\cos^2(x) - 8\sin^2(x) & f^{(4)}(\pi) &= 8. \end{aligned}$$

So we have

$$T_4(x, \pi) = 1 - (x-\pi)^2 + \frac{1}{3}(x-\pi)^4.$$

(b) Using series we already know, write down a formula for the (infinite) Taylor series for $(1-2x)^{-3}$, and then write down the degree-four polynomial explicitly.

Solution: We can take this from the binomial series. So we have

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \binom{-3}{n} (-2x)^n = \sum_{n=0}^{\infty} \binom{-3}{n} (-2)^n x^n \\ T_4(x, 0) &= 1 + (-2)\frac{-3}{1}x + 4\frac{12}{2}x^2 + (-8)\frac{-60}{6}x^3 + (16)\frac{360}{24}x^4 \\ &= 1 + 6x + 24x^2 + 80x^3 + 240x^4 \end{aligned}$$

(c) If $f(x) = \sum_{n=0}^{\infty} 2^n n^3 (x-2)^n$, compute $\frac{d}{dx} f(x)$ and $\int f(x)\,dx$.

Solution:

$$\begin{aligned} \frac{d}{dx} f(x) &= \sum_{n=0}^{\infty} 2^n n^4 (x-2)^{n-1} \\ \text{(or better)} &= \sum_{n=1}^{\infty} 2^n n^4 (x-2)^{n-1} \\ \int f(x)\,dx &= \sum_{n=0}^{\infty} \frac{2^n n^3}{n+1} (x-2)^{n+1} + C \\ \text{(or)} &= \sum_{n=1}^{\infty} \frac{2^{n-1}(n-1)^3}{n} (x-2)^n + C. \end{aligned}$$

S9: Applications of Taylor Series

(a) Use a degree-three Taylor polynomial to estimate $\sqrt{1.2}$.

Solution:

$$\begin{aligned}\sqrt{1+x} &= 1 + \frac{1}{2}x + \frac{(1/2)(-1/2)}{2!}x^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}x^3 \\ &= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} \\ \sqrt{1.2} &= 1 + \frac{.2}{2} - \frac{.04}{8} + \frac{.008}{16} = 1 + .1 - .005 + .0005 = 1.0955.\end{aligned}$$

(b) Use a Taylor series to compute $\lim_{x\to 0} \frac{xe^{x^3} - x - x^4}{x^7} =$

Solution:

$$\begin{aligned}\lim_{x\to 0} \frac{xe^{x^3} - x - x^4}{x^7} &= \lim_{x\to 0} \frac{(x + x^4 + x^7/2 + x^{10}/3! + \dots) - x + x^4}{x^7} \\ &= \lim_{x\to 0} \frac{x^7/2 + x^{10}/3! + \dots}{x^7} \\ &= \lim_{x\to 0} \frac{1}{2} - \frac{x^3}{3!} + \cdots = \frac{1}{2}.\end{aligned}$$

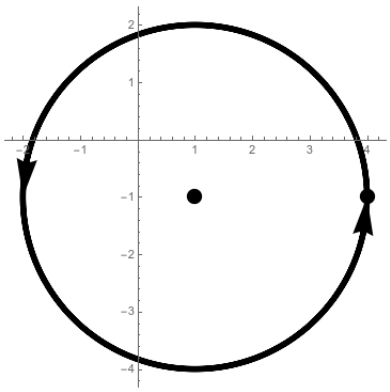
(c) Using series, compute $\int_0^\pi 2x\cos(x^5)\,dx$.

Solution:

$$\begin{aligned}\cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} \\ \cos(x^5) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{10n} \\ 2x\cos(x^5) &= \sum_{n=0}^{\infty} \frac{2(-1)^n}{(2n)!}x^{10n+1} \\ \int 2x\cos(x^5)\,dx &= \sum_{n=0}^{\infty} \frac{2(-1)^n}{(2n)!(10n+2)}x^{10n+2} + C \\ \int_0^\pi 2x\cos(x^5)\,dx &= \sum_{n=0}^{\infty} \frac{2(-1)^n}{(2n)!(10n+2)}\pi^{10n+2}\end{aligned}$$

S10: Parametrization

(a) Find a parametrization for the circle of radius 3 centered at $(1, -1)$, starting at $(4, -1)$ and going **counterclockwise twice** around the circle.



Solution: A circle has parametrization $\vec{r}(t) = (\cos(t), \sin(t))$. To make it radius 3 we multiply by 3, and then we shift it over to have center $(1, -1)$, and get

$$\vec{r}(t) = 3\cos(t) + 1, 3\sin(t) - 1.$$

In order to make it go around twice, we have $0 \leq t \leq 4\pi$. Alternatively, we could have $0 \leq 2 \leq 2\pi$ and use the equations

$$\vec{r}(t) = 3\cos(2t) + 1, 3\sin(2t) - 1.$$

(There are a bunch of other options that also work but these are the two most obvious to me.)

(b) Find a parametrization of the ellipse $x^2/4 + y^2 = 1$. (Hint: what are the x and y intercepts?)

Solution: An ellipse is like a circle, but it's wider in one direction than the other. In particular, this ellipse goes through $(2, 0), (0, 1), (-2, 0), (0, -1)$. So we can take

$$x = 2\cos(t)$$
$$y = sin(t).$$

There are also lots of other options; another would be

$$x = 2\sin(t)$$
$$y = \cos(t).$$

This would start at a different point and go clockwise instead of counterclockwise, but still cover the entire ellipse.

(c) Find a parametric equation for the line tangent to the curve $x = 1 + \sqrt{t}, y = t^3$ at the point $(2, 1)$.

Solution: We have $x'(t) = \frac{1}{2\sqrt{t}}$ and $y'(t) = 3t^2$. This point happens at $t = 1$, so we have $x'(1) = \frac{1}{2}$ and $y'(1) = 3$. Then we get

$$T(t) = (2, 1) + (1/2, 3)(t - 1) = (3/2 + t/2, -2 + 3t)$$

or instead we could have

$$T(t) = (2, 1) + (1/2, 3)t = (2 + t/2, 1 + 3t).$$