Math 1232: Single-Variable Calculus 2
George Washington University   Spring 2024
Recitation 12

Jay Daigle

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Problem 1. Let's find the Taylor series of $f(x) = e^x$ centered at $a = 1$.

(a) Compute f', f'', f'''. Find a formula for $f^{(n)}(x)$.

(b) Give a formula for $T_f(x, 1)$.

(c) We want to know if $f(x) = T_f(x, 1)$. Find a formula for $R_k(x, 1)$. Can you show this goes to 0 as k goes to infinity?

(d) We already have another power series for f:

$$T_f(x, 0) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

You should have a different power series; but can you convince yourself it *should* give the same function? (What happens if you plug $x - 1$ into this series?)

Solution:

(a) $f^{(n)}(x) = e^x$.

(b) We know that $e^1 = e$, so we have

$$T_f(x, 1) = \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n.$$

(c) We compute that

$$R_k(x,1) = \frac{e^z}{(k+1)!}(x-1)^{k+1}.$$

If $x > 0$, we know that $e^z < e^x$; if $x < 0$ then $e^z < 1$. Either way, we can fix x, and as k goes to infinity this goes to 0. So f is equal to its Taylor series centered at 1.

(d) We see that

$$T_f(x-1,0) = \sum_{n=0}^{\infty} \frac{1}{n!}(x-1)^n$$

which is almost the same as $T_f(x,1)$. This makes sense: from properties of e^x, we know that

$$\begin{aligned} e^x = e \cdot e^{x-1} &= e \cdot \sum_{n=0}^{\infty} \frac{1}{n!}(x-1)^n \\ &= \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n = T_f(x,1). \end{aligned}$$

Problem 2. Let's do something silly, and compute the Taylor series of a polynomial.

(a) Let $f(x) = x^3 + 3x^2 + 1$. Find the Taylor series centered at zero. Was that what you expected?

(b) Now find the Taylor series centered at 2. Do you get the same thing? What's useful about this?

Solution:

(a) Let $f(x) = x^3 + 3x^2 + 1$. Then we have $f'(x) = 3x^2 + 6x$, $f''(x) = 6x + 6$, $f'''(x) = 6$, and $f^{(n)}(x) = 0$ for $n > 3$. Thus the Taylor series centered at 0 is

$$\begin{aligned} T_f(x,0) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 \\ &= 1 + 0x + \frac{6}{2}x^2 + \frac{6}{6}x^3 = 1 + 3x^2 + x^3. \end{aligned}$$

Hopefully this is what you expected.

If we take the Taylor series centered at 2, for instance, we have

$$\begin{aligned} T_f(x,2) &= f(2) + f'(2)x + \frac{f''(2)}{2}x^2 + \frac{f'''(2)}{6} \\ &= 21 + 24(x-2) + \frac{18}{2}(x-2)^2 + \frac{6}{6}(x-2)^3 \\ &= 21 + 24(x-2) + 9(x-2)^2 + (x-2)^3. \end{aligned}$$

If you multiply this out you will get your original polynomial back, so this is the same thing. But sometimes it is very useful to have a polynomial expressed in terms of $x-2$, say, instead of in terms of x. This is the easiest way I know of to rewrite your polynomial that way.

Problem 3. Back in section 2 we talked about the bell curve function $p(x) = e^{-x^2}$. (Technically we should be talking about $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ but that's annoying and doesn't change the details enough to be interesting.)

(a) Find a power series for $p(x)$ centered at zero. (This should not require any real calculations.)

(b) Find an antiderivative for $p(x)$, using power series.

(c) Write down a series that computes $\int_0^1 p(x)\,dx$.

(d) Add up the first three or four terms of this series. What do you get? Can you estimate the error in this calculation?

Solution:

(a)

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

(b)

$$\int e^{-x^2}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(n!)} + C.$$

(c)

$$\int_0^1 e^{-x^2}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(n!)}\Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(n!)}$$

(d)

$$\int_0^1 e^{-x^2}\,dx \approx \sum_{n=0}^{3} \frac{(-1)^n}{(2n+1)(n!)} = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} = \frac{26}{35} \approx 0.74.$$

This is an alternating series, so the error has to be smaller than the next term $\frac{1}{9 \cdot 24} = \frac{1}{216} \approx .005$. So this is correct to two decimal places.

(In fact the true integral is 0.746824.)

Problem 4. In class we worked out a Taylor series for $g(x) = \ln(x)$ centered at $a = 1$:

$$T_g(x,1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n.$$

But is this actually equal to $g(x)$?

(a) Write down a formula for $R_k(x,1)$.

(b) Compute $T_5(2,1)$. Can you estimate the error?

(c) Compute $T_5(1.5,1)$. Can you estimate the error?

(d) Compute $T_5(0,1)$. Can you estimate the error here?

(e) What would you need to assume to show this goes to zero as k goes to infinity? Does that makes sense?

Solution:

(a) We know

$$R_k(x,1) = \frac{f^{(k+1)}(z)}{(k+1)!}(x-1)^{k+1}.$$

We worked out in class that $f^{(k+1)}(x) = (-1)^k k! x^{-k-1}$, which tells us that $f^{(k+1)}(z) = \frac{(-1)^k k!}{z^{k+1}}$. Thus we have

$$R_k(x,1) = \frac{(-1)^k k!}{(k+1)! z^{k+1}} x^{k+1} = \frac{(-1)^k}{(k+1)} \frac{(x-1)^{k+1}}{z^{k+1}}$$

where z is somewhere in between 1 and x.

(b) We have

$$T_5(x,1) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5$$
$$T_5(2,1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = \frac{47}{60} \approx .7833.$$

We have

$$R_5(x,1) = \frac{(-1)^5}{6}\frac{(x-1)^6}{z^6}$$
$$R_5(2,1) = \frac{-1}{6}\cdot\frac{1}{z^6}.$$

We need z to between 1 and 2, so this is maximized at $\frac{-1}{6}$. So the error is at most $1/6$.

(c) We have

$$T_5(x,1) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5$$
$$T_5(1.5,1) = .5 - \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{3}\cdot\frac{1}{8} - \frac{1}{4}\cdot\frac{1}{16} + \frac{1}{5}\cdot\frac{1}{32}$$
$$= \frac{391}{960} \approx .4073.$$

We have

$$R_5(x,1) = \frac{(-1)^5}{6}\frac{(x-1)^6}{z^6}$$
$$R_5(1.5,1) = \frac{-1}{6}\cdot\frac{.5^6}{z^6} = \frac{-1}{384z^6}$$

We need z to between 1 and 1.5, so this is maximized at $\frac{-1}{385}$. So the error is at most $1/384$. This is pretty good!

(d) We have

$$T_5(x,1) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5$$
$$T_5(0,1) = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} = \frac{-135}{60} \approx -2.28333.$$

We have

$$R_5(x,1) = \frac{(-1)^5}{6}\frac{(x-1)^6}{z^6}$$
$$R_5(0,1) = \frac{-1}{6}\cdot\frac{1}{z^6} = \frac{-1}{6z^6}.$$

This looks like our answer from part (b), but it really isn't. this time we need z between 0 and 1, so this is maximized when z is as small as possible—this error bound could be extremely big.

And that makes sense, because the power series doesn't converge here. And the function goes to infinity here. Our error "really is" infinity—we can't get a good bound.

(e) We need $x-1$ to be small. So in particular, if we assume that x is between 1 and 2, then we have

$$\begin{aligned} |R_k(x,1)| &= \left| \frac{(-1)^k}{(k+1)} \frac{(x-1)^{k+1}}{z^{k+1}} \right| \\ &\leq \frac{1}{k+1} \frac{1}{z^{k+1}} \end{aligned}$$

and since z has to be between 1 and 2, this is $\leq \frac{1}{k+1}$, and the error tends to zero as k tends to infinity.

If $x > 2$, we can't show this error is controlled. And that makes sense—because we know the series doesn't converge for $x > 2$, from our theory of geometric series.