Math 1232 Spring 2024
Single-Variable Calculus 2 Section 12
Mastery Quiz 13
Due Tuesday, April 30

This week's optional mastery quiz has three topics. You may not need to submit any. If you have a 4/4 on M4, or a 2/2 on S9 or S10, you don't need to submit them.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 4: Taylor Series
- Secondary Topic 9: Applications of Taylor Series
- Secondary Topic 10: Parametrization

Name:

Recitation Section:

M4: Taylor Series

(a) Using series we already know, write down a formula for the (infinite) Taylor series for $x^3 e^{x^5/4}$, and then write down the first four non-zero terms of this series.

Solution:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{1}{n!} x^n \\ e^{x^5/4} &= \sum_{n=0}^{\infty} \frac{1}{n!} (x^5/4)^n = \sum_{n=0} \frac{1}{n! \cdot 4^n} x^{5n} \\ x^3 e^{x^5/4} &= \sum_{n=0} \frac{1}{n! \cdot 4^n} x^{5n+3} \end{aligned}$$

The first four non-zero terms are

$$x^3 + \frac{1}{4} x^8 + \frac{1}{32} x^{13} + \frac{1}{6 \cdot 64} x^{18}.$$

(Note: this is *not* T_3 or T_4. It's T_{18}!)

(b) Find an upper bound for the error if you use $T_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$ to approximate $g(x) = \ln(1 + x)$ at $x = .5$.

Solution: We can use the remainder theorem. We have

$$|R_3(x)| = \left| \frac{f^{(4)}(z)}{4!} x^4 \right| = \left| \frac{3!}{(1+z)^4 4!} x^4 \right| = \left| \frac{x^4}{4(1+z)^4} \right|$$

We know that z is between 0 and .5, so we have

$$\begin{aligned} \frac{1}{(1+z)^4} &\leq \frac{1}{1^4} = 1 \\ |R_3(.5)| &= \left| \frac{.5^4}{4(1+z)^4} \right| \leq \frac{.5^4}{4} \cdot 1 = \frac{1}{64}. \end{aligned}$$

(c) Using series we already know, write down a formula for the (infinite) Taylor series for $x(8 + x)^{5/3}$, and then write down the degree-four polynomial explicitly.

Solution: We can take this from the binomial series for $(1+x)^\alpha$. So we have

$$\begin{aligned}
(1+x/8)^{5/3} &= \sum_{n=0}^{\infty} \binom{5/3}{n} (x/8)^n \\
(8+x)^{5/3} &= 32(1+x/8)^{5/3} = \sum_{n=0}^{\infty} \binom{5/3}{n} 2^{5-3n} x^n \\
x(8+x)^{5/3} &= \sum_{n=0}^{\infty} \binom{5/3}{n} 2^{5-3n} x^{n+1} \\
T_4(x,0) &= 32x + {}^5\!/_3 \cdot 4x^2 + \frac{({}^5\!/_3)({}^2\!/_3)}{2} \cdot \frac{1}{2} x^3 + \frac{({}^5\!/_3)({}^2\!/_3)(-{}^1\!/_3)}{6} \frac{1}{16} x^4 \\
&= 32x + \frac{20}{3}x^2 + \frac{5}{18}x^3 - \frac{5}{1296}x^4.
\end{aligned}$$

S9: Applications of Taylor Series

(a) Use a Taylor series to compute $\lim_{x\to 0} \frac{\cos(x^2) - 1 + x^4/2}{x^8} =$

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos(x^2) - 1 + x^4/2}{x^8} &= \lim_{x\to 0} \frac{(1 - x^4/2 + x^8/4! - x^{12}/6! + \dots) - 1 + x^4/2}{x^8} \\
&= \lim_{x\to 0} \frac{x^8/4! - x^{12}/6! + \dots}{x^8} \\
&= \lim_{x\to 0} \frac{1}{4!} - \frac{x^4}{6!} + \cdots = \frac{1}{24}.
\end{aligned}$$

(b) Use a degree-five Taylor polynomial to estimate arctan(.1).

Solution: We have

$$\begin{aligned}
\arctan(x) &\approx x - x^3/3 + x^5/5 \\
\arctan(.1) &\approx .1 - (.1)^3/3 + (.1)^5/5 = .1 - .00033\cdots + .000002 = .09966866\dots.
\end{aligned}$$

(c) Use a degree-three Taylor polynomial to estimate $(1.1)^{3.1}$.

Solution:

$$\begin{aligned}
(1.1)^{3.1} &\approx 1 + 3.1x + \frac{3.1 \cdot 2.1}{1 \cdot 2}x^2 + \frac{3.1 \cdot 2.1 \cdot 1.1}{1 \cdot 2 \cdot 3}x^3 \\
&= 1 + 3.1x + 3.255x^2 + 1.1935x^3 \\
(1.1)^{3.1} &\approx 1 + 3.1(.1) + 3.255(.1)^2 + 1.1935(.1)^3 = 1 + .31 + .03255 + .0011935 = 1.3437435.
\end{aligned}$$

S10: Parametrization

(a) Find the length of the curve parametrized by $x = e^t - t, y = 4e^{t/2}$ for $0 \le t \le 2$.

Solution: We have $x'(t) = e^t - 1$ and $y'(t) = 2e^{t/2}$, so the arc length is

$$\begin{aligned}
L &= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\
&= \int_0^2 \sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\, dt \\
&= \int_0^2 \sqrt{e^{2t} - 2e^2 + 1 + 4e^t}\, dt = \int_0^2 \sqrt{e^{2t} + 2e^t + 1}\, dt \\
&= \int_0^2 e^t + 1\, dt = e^t + t\Big|_0^2 = e^2 + 2 - 1 = e^2 + 1.
\end{aligned}$$

(b) Find an equation of the line tangent to the curve $x = \cos^3(t), y = \sin^3(t)$ at the point $(1/8, -3\sqrt{3}/8)$.

Solution: We have $x'(t) = -3\cos^2(t)\sin(t)$ and $y'(t) = 3\sin^2(t)\cos(t)$. This point happens at $t = -\pi/3$, so we have

$$\begin{aligned}
x'(-\pi/3) &= 3\sqrt{3}/8 \\
y'(-\pi/3) &= 9/8 \\
\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{9/8}{3\sqrt{3}/8} = \sqrt{3} \\
y + \frac{3\sqrt{3}}{8} &= \sqrt{3}(x - 1/8).
\end{aligned}$$