Math 1232: Single-Variable Calculus 2
George Washington University   Spring 2024
Recitation 2

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Problem 1. (a) Compute $\log_3(6) + \log_3(9/2)$.

(b) Compute $\log_4(8) - \log_4(2)$.

(c) Rewrite the expression $\log_5(15) + \log_5(75) - \log_5(12)$ as an integer plus a logarithm.

(d) Solve $e^{5-3s} = 10$.

Solution:

(a) $\log_3(6) + \log_3(9/2) = \log_3(6 \cdot 9/2) = \log_3(27) = 3$.

(b) $\log_4(8) - \log_4(2) = \log_4(4) = 1$.

Alternatively $\log_4(8) - \log_4(2) = 1.5 - .5 = 14$.

(c)

$$\log_5(15) + \log_5(75) - \log_5(12) = \log_5(15 \cdot 75/12)$$
$$= \log_5\left(\frac{3}{4} \cdot 125\right) \quad = \log_5(125) + \log_5(3/4) = 3 + \log_5(3/4).$$

You could also write this as $3 + \log_5(3) - \log_5(4)$ if you want.

(d) The problem here is that there's a variable in the exponent. To deal with difficult exponents, we take a logarithm. Then we see that $5 - 3x = \ln 10$ and so $x = \frac{5-\ln 10}{3}$.

Problem 2. Compute the derivative of $(x+1)^{\sqrt{x}}$.

Solution: We can't do this from the derivative rules we already have. But we can use logarithms!

$$\begin{aligned}
y &= (x+1)^{\sqrt{x}} \\
\ln|y| &= \ln\left|(x+1)^{\sqrt{x}}\right| = \sqrt{x}\ln|x+1| \\
\frac{y'}{y} &= \frac{1}{2\sqrt{x}}\ln|x+1| + \frac{\sqrt{x}}{|x+1|} \\
y' &= y\left(\frac{1}{2\sqrt{x}}\ln|x+1| + \frac{\sqrt{x}}{|x+1|}\right) \\
&= (x+1)^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\ln|x+1| + \frac{\sqrt{x}}{|x+1|}\right).
\end{aligned}$$

Problem 3 (Bonus)**.** Use logarithmic differentiation to compute $\frac{d}{dx}\frac{x^3\sqrt{x^2-5}}{(x+4)^3}$.

Solution:

$$\begin{aligned}
y &= \frac{x^3\sqrt{x-5}}{(x+4)^3} \\
\ln|y| &= \ln\left|\frac{x^3\sqrt{x^2-5}}{(x+4)^3}\right| \\
&= 3\ln|x| + \frac{1}{2}\ln|x^2-5| - 3\ln|x+4| \\
\frac{y'}{y} &= \frac{3}{x} + \frac{1}{2}\frac{2x}{x^2-5} - \frac{3}{x+4} \\
y' &= y\left(\frac{3}{x} + \frac{1}{2}\frac{2x}{x^2-5} - \frac{3}{x+4}\right) \\
&= \frac{x^3\sqrt{x^2-5}}{(x+4)^3}\left(\frac{3}{x} + \frac{x}{x^2-5} - \frac{3}{x+4}\right).
\end{aligned}$$

If we want we can even simplify this to

$$y' = \frac{3x^2\sqrt{x^2-5}}{(x+4)^3} + \frac{x^4}{(x+4)^3\sqrt{x^2-5}} - \frac{3x^3\sqrt{x^2-5}}{(x+4)^4}$$

Problem 4. Consider the integral $\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\,dx$.

(a) We're going to have to do a u-substitution here. What u looks like it should work?

(b) What do we need to change the bounds to when we do the u-substitution?

(c) Compute $\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\,dx$.

(d) Now try computing $\int \frac{1}{x\sqrt{\ln x}}\,dx$ to get the antiderivative.

(e) Now plug e^4 and e in to your antiderivative. What do you notice? How is this related to part (c)?

Solution:

(a) We take $u = \ln(x)$, and $du = \frac{dx}{x}$. This seems plausible because $\ln(x)$ is on the inside of a function.

(b) $\ln(e) = 1$ and $\ln(e^4) = 4$, so we have to integrate from 1 to 4.

(c)
$$\int_e^{e^4} \frac{1}{x\sqrt{\ln x}} dx = \int_1^4 \frac{1}{\sqrt{u}}\,du = 2\sqrt{u}\Big|_1^4 = 4 - 2 = 2.$$

(d)
$$\int \frac{1}{x\sqrt{\ln x}} dx = \int \frac{1}{\sqrt{u}}\,du = 2\sqrt{u} + C = 2\sqrt{\ln(x)} + C.$$

(e) We get $2\sqrt{\ln(e^4)} = 2\sqrt{4} = 4$ and $2\sqrt{\ln(e)} = 2\sqrt{1} = 2$, which are the same numbers we got before. And specifically, you see we get 4 and 1 as intermediate answers here—these are the same numbers we got by doing the change of bounds.

Problem 5. Compute the following integrals.

(a) $\int e^x \cos(1 + e^x)\,dx$.

(b) $\int \frac{\ln(x)}{x}\,dx$.

Solution:

(a) Take $u = 1 + e^x$ so $du = e^x dx$. Then
$$\int e^x \cos(1 + e^x)\,dx = \int \cos(u)\,du = \sin(u) + C = \sin(1 + e^x) + C.$$

(b) This one looks tricky, and you might have to mess around with it a bit to see, and try different things. But if we take $u = \ln(x)$ so that $du = \frac{1}{x}\,dx$, we see this is
$$\int u\,du = \frac{u^2}{2} + C = \frac{(\ln|x|)^2}{2} + C.$$

Problem 6 (Challenge)**.** Compute $\int \frac{dx}{1 + e^x}$.

Solution: This problem becomes much easier if we multiply the top and bottom by e^{-x}. Then we have $\int \frac{e^{-x}}{e^{-x}+1}\,dx$. Set $u = e^{-x}$ so that $du = -e^{-x}\,dx$ and we have

$$\int \frac{e^{-x}}{e^{-x}+1}\,dx = -\int \frac{du}{1+u} = -\ln(1+u) = -\ln(1+e^{-x}).$$

Alternatively, we can take $u = e^x, du = e^x\,dx$, and have

$$\int \frac{dx}{1+e^x} = \int \frac{du}{u(u+1)}.$$

Again nonobviously, we write

$$\begin{aligned}\int \frac{du}{u(u+1)} = \int \frac{1+u-u}{u(u+1)}\,du &= \int \frac{1+u}{u(u+1)}\,du - \int \frac{u}{u(u+1)}\,du \\ &= \int \frac{du}{u} - \int \frac{du}{u+1} \\ &= \ln(u) - \ln(u+1) = \ln(e^x) - \ln(e^x+1).\end{aligned}$$

Using properties of logs, you can check that this is the same as the previous answer. Or, if you prefer, you can write $x - \ln(e^x+1)$.