# Math 1232 Spring 2024
# Single-Variable Calculus 2 Section 12
# Mastery Quiz 4
# Due Tuesday, February 13

This week's mastery quiz has three topics. Everyone should submit work on topic M2. If you have a 2/2 on S2, or a 4/4 on M1, you don't need to submit them again. (Check Blackboard!)

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

**Topics on This Quiz**

- Major Topic 1: Calculus of Transcendental Functions
- Major Topic 2: Advanced Integration Techniques
- Secondary Topic 2: L'Hospital's Rule

## Name:

## Recitation Section:

# M1: Calculus of Invertible Functions

(a) Compute $\dfrac{d}{dx}\cos(x)^{\sin(x)}$

**Solution:**

$$\begin{aligned} y &= \cos(x)^{\sin(x)} \\ ln|y| &= \sin(x)\ln|\cos(x)| \\ y'/y &= \cos(x)\ln|\cos(x)| + \sin(x)\frac{-\sin(x)}{\cos(x)} \\ y' &= \cos(x)^{\sin(x)}\left(\cos(x)\ln|\cos(x)| - \frac{\sin^2(x)}{\cos(x)}\right) \end{aligned}$$

(b) $\displaystyle\int 3^x(4+3^x)^3\,dx =$

**Solution:** Set $u = 4 + 3^x$ so that $du = 3^x\ln(3)\,dx$. Then

$$\begin{aligned} \int 3^x(4+3^x)^3\,dx &= \int \frac{1}{\ln(3)}u^3\,du \\ &= \frac{u^4}{4\ln(3)} + C = \frac{(4+3^x)^4}{4\ln(3)} + C \end{aligned}$$

(c) $\displaystyle\int \frac{1}{x\sqrt{4-\ln(x)^2}}\,dx =$

**Solution:** We can set $u = \ln(x)/2$ so $du = \frac{1}{2x}\,dx$. Then

$$\begin{aligned} \int \frac{1}{x\sqrt{4-\ln(x)^2}}\,dx &= \int \frac{2}{\sqrt{4-4u^2}}\,du = \int \frac{1}{\sqrt{1-u^2}}\,du \\ &= \arcsin(u) + C = \arcsin(\ln(x)/2) + C. \end{aligned}$$

# M2: Advanced Integration Techniques

(a) $\displaystyle\int \sin(2x)\cos(3x)\,dx =$

(Please do *not* use any product-of-trig-function identities we haven't discussed in class.)

**Solution:**

$$\begin{aligned}
\int \sin(2x)\cos(3x)\,dx &= \frac{1}{3}\sin(2x)\sin(3x) - \int \frac{2}{3}\cos(2x)\sin(3x)\,dx \\
\int \cos(2x)\sin(3x)\,dx &= -\frac{1}{3}\cos(2x)\cos(3x) - \int \frac{2}{3}\sin(2x)\cos(3x) \\
\int \sin(2x)\cos(3x)\,dx &= \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x) + \frac{4}{9}\int \sin(2x)\cos(3x)\,dx \\
\frac{5}{9}\int \sin(2x)\cos(3x)\,dx &= \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x)\big(+C\big) \\
\int \sin(2x)\cos(3x)\,dx &= \frac{3}{5}\sin(2x)\sin(3x) + \frac{2}{5}\cos(2x)\cos(3x) + C.
\end{aligned}$$

(b) $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}} =$

**Solution:** We can take $x = 2\sin(\theta)$, so that $dx = 2\cos(\theta)\,d\theta$. Then we get

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{4-x^2}} &= \int \frac{2\cos(\theta)\,d\theta}{4\sin^2(\theta)\sqrt{4-4\sin^2(\theta)}} \\
&= \int \frac{2\cos(\theta)\,d\theta}{4\sin^2(\theta)\sqrt{4\cos^2(\theta)}} \\
&= \int \frac{2\cos(\theta)\,d\theta}{4\sin^2(\theta)\cdot 2\cos(\theta)} \\
&= \int \frac{d\theta}{4\sin^2(\theta)} = \int \frac{1}{4}\csc^2(\theta)\,d\theta \\
&= -\frac{1}{4}\cot(\theta) + C.
\end{aligned}$$

Now we need to substitute our $x$ back in. We know that $\sin(\theta) = x/2$, so we can construct a right triangle with opposite side $x$, hypotenuse 2, and thus adjacent side $\sqrt{4-x^2}$. (Note that this is the term that showed up in the original integral!)

Then $\cot(\theta) = \frac{\text{adjacent}}{\text{opposite}} = \frac{\sqrt{4-x^2}}{x}$, and so we have

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\cot(\theta) + C = -\frac{\sqrt{4-x^2}}{4x} + C.$$

(c) $\displaystyle\int_0^{\pi/6} \sec^3(2t)\tan(2t)\,dt =$

**Solution:** We're going to take $u = \sec(2t)$ so that $du = 2\sec(2t)\tan(2t)\,dt$. We compute $u(0) = 1$ and $u(\pi/6) = \sec(\pi/3) = 2$. Then

$$\begin{aligned}\int_0^{\pi/6} \sec^3(2t)\tan(2t)\,dt &= \int_1^2 \frac{1}{2}u^2\,du \\ &= \frac{u^3}{6}\Big|_1^2 = \frac{8}{6} - \frac{1}{6} = \frac{7}{6}.\end{aligned}$$

# S2: L'Hospital's Rule

(a) $\displaystyle\lim_{x\to 2} \frac{e^{(x^2-4)} - x + 1}{x-2} =$

**Solution:** The limit of the top and bottom are both 0, we can use L'Hospital's rule.

$$\lim_{x\to 2} \frac{e^{x^2-4} - x + 1^{\nearrow 0}}{x-2_{\searrow 0}} \stackrel{\text{L'H}}{=} \lim_{x\to 2} \frac{2xe^{x^2-4} - 1}{1} = 3.$$

(b) $\displaystyle\lim_{x\to 1} \frac{\ln(x)}{\arcsin(2x-2)} =$

**Solution:** The top and bottom both approach 0, so we can use L'Hospital's Rule:

$$\begin{aligned}\lim_{x\to 1} \frac{\ln(x)^{\nearrow 0}}{\arcsin(x-1)_{\searrow 0}} &\stackrel{\text{L'H}}{=} \lim_{x\to 1} \frac{1/x}{\frac{2}{\sqrt{1-(2x-2)^2}}} \\ &= \lim_{x\to 1} \frac{\sqrt{1-(x-1)^2}}{2x} = \frac{1}{2}.\end{aligned}$$

(c) $\displaystyle\lim_{x\to\infty} x^{\ln(3)/(2+\ln(x))} =$

**Solution:**

$$\begin{aligned}\ln(y) &= \frac{\ln(3)\ln(x)}{2+\ln(x)} \\ \lim_{x\to\infty} \frac{\ln(3)\ln(x)}{2+\ln(x)} &= \lim_{x\to\infty} \frac{\ln(3)/x}{1/x} = \ln(3) \\ \lim_{x\to\infty} y &= e^{\ln(3)} = 3.\end{aligned}$$