Math 1232: Single-Variable Calculus 2
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Recitation 4

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Problem 1. (a) We want to compute $\int x^2 e^{-3x}\, dx$. Why do we want to use integration by parts? What should be our u and dv, and why?

(b) Compute the integral.

(c) Now we want to compute $\int \cos(3x) e^{2x}\, dx$. Why do we want to use integration by parts? What should be our u and dv, and why? When we need to make another choice, what forces us to make that choice?

(d) Compute the integral.

Solution:

(a) We see a product of two functions that don't really have any useful interactions, so we want to use integration by parts. x^2 gets simpler when we differentiate, and e^{-3x} doesn't, so we're going to want to differentiate the x^2 term. So we set $u = x^2$ and $dv = e^{-3x}\, dx$, so that $du = 2x$ and $v = \frac{-1}{3} e^{-3x}$.

(b)

$$\begin{aligned}
\int x^2 e^{-3x}\,dx &= x^2\frac{-1}{3}e^{-3x} - \int 2x\frac{-1}{3}e^{-3x}\,dx \\
&= -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\int xe^{-3x}\,dx \\
&= -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\left(x\frac{-1}{3}e^{-3x} - \int \frac{-1}{3}e^{-3x}\,dx\right) \\
&= -\frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3x} + \frac{2}{9}\int e^{-3x}\,dx \\
&= -\frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C.
\end{aligned}$$

(c) We again see a product of two functions that don't have any useful interactions. Here it's really hard to see which one would be better to differentiate or integrate! I'm going to integrate the cosine and differentiate the e^{2x} but it doesn't matter that much. So I take $u = e^{2x}$ and $dv = \cos(3x)\,dx$, which gives $du = 2e^{2x}\,dx$ and $v = \frac{1}{3}\sin(3x)$.

(d)

$$\begin{aligned}
\int \cos(3x)e^{2x}\,dx &= \frac{1}{3}\sin(3x)e^{2x} - \int \frac{1}{3}\sin(3x)\cdot 2e^{2x}\,dx \\
&= \frac{1}{3}\sin(3x)e^{2x} - \frac{2}{3}\int \sin(3x)e^{2x}\,dx.
\end{aligned}$$

Here we need to do integration by parts again, but we *have to* take e^{2x} to be our u this time, since we did last time. That means $dv = \sin(3x)\,dx$ and $v = \frac{-1}{3}\cos(3x)$. And we get

$$\begin{aligned}
\int \cos(3x)e^{2x}\,dx &= \frac{1}{3}\sin(3x)e^{2x} - \frac{2}{3}\int \sin(3x)e^{2x}\,dx \\
&= \frac{1}{3}\sin(3x)e^{2x} - \frac{2}{3}\left(\frac{-1}{3}\cos(3x)e^{2x} - \int \frac{-1}{3}\cos(3x)\cdot 2e^{2x}\,dx\right) \\
&= \frac{1}{3}\sin(3x)e^{2x} + \frac{2}{9}\cos(3x)e^{2x} - \frac{4}{9}\int \cos(3x)e^{2x}\,dx.
\end{aligned}$$

Now we need to use our big trick, because this integral is the same one we started with. So we get

$$\begin{aligned}
\int \cos(3x)e^{2x}\,dx &= \frac{1}{3}\sin(3x)e^{2x} + \frac{2}{9}\cos(3x)e^{2x} - \frac{4}{9}\int \cos(3x)e^{2x}\,dx \\
\frac{13}{9}\int \cos(3x)e^{2x}\,dx &= \frac{1}{3}\sin(3x)e^{2x} + \frac{2}{9}\cos(3x)e^{2x} + C \\
\int \cos(3x)e^{2x}\,dx &= \frac{9}{13}\left(\frac{1}{3}\sin(3x)e^{2x} + \frac{2}{9}\cos(3x)e^{2x}\right) + C \\
&= \frac{3}{13}\sin(3x)e^{2x} + \frac{2}{13}\cos(3x)e^{2x} + C.
\end{aligned}$$

Problem 2. Compute $\int \arctan(x)\,dx$.

Solution: We know the derivative of arctan but not the integral. But integration by parts lets us convert! We can take $u = \arctan(x)$ and $dv = dx$, so then $du = \frac{dx}{1+x^2}$ and $v = x$. Then we have

$$\begin{aligned}
\int \arctan(x)\,dx &= x\arctan(x) - \int x\frac{dx}{1+x^2} \\
&= x\arctan(x) - \int \frac{x}{u}\cdot\frac{du}{2x} && (u = 1+x^2) \\
&= x\arctan(x) - \frac{1}{2}\int \frac{1}{u}\,du \\
&= x\arctan(x) - \ln|u| + C = x\arctan(x) - \ln|x^2+1| + C.
\end{aligned}$$

(Again we see the close relationship between inverse trig and logarithms.)

Problem 3. Compute $\int \sin^6(x)\,dx$.

Solution: By the double angle formula, we have

$$\begin{aligned}
\int \sin^6(x)\,dx &= \int \left(\frac{1-\cos(2x)}{2}\right)^3 dx \\
&= \frac{1}{8}\int (1-\cos(2x))^3\ dx \\
&= \frac{1}{8}\int 1 - 3\cos(2x) + 3\cos^2(2x) - \cos^3(2x)\,dx \\
&= \frac{1}{8}\int 1 - 3\cos(2x) + 3\frac{1+\cos(4x)}{2} - (1-\sin^2(2x))\cos(2x)\,dx \\
&= \frac{1}{8}\int 1 - 3\cos(2x) + \frac{3}{2} + \frac{3}{2}\cos(4x) - \cos(2x) + \sin^2(2x)\cos(2x)\,dx \\
&= \frac{1}{8}\int \frac{5}{2} - 4\cos(2x) + \frac{3}{2}\cos(4x) + \sin^2(2x)\cos(2x)\,dx \\
&= \frac{1}{8}\left(\frac{5x}{2} - 2\sin(2x) + \frac{3}{8}\sin(4x) + \frac{1}{6}\sin^3(2x)\right) + C.
\end{aligned}$$

Problem 4. Compute $\displaystyle\int \sec^6(x)\tan^5(x)\,dx$ with two different approaches. Do you get the same answer either way?

Solution: One option is to reduce until we have two secant terms. Then we can set $u = \tan(x)$ and $du = \sec^2(x)\,dx$. We compute

$$\begin{aligned}\int \sec^6(x)\tan^5(x)\,dx &= \int \sec^2(x)(1+\tan^2(x))^2\tan^5(x)\,dx \\ &= \int \sec^2(x)\tan^5(x) + 2\sec^2(x)\tan^7(x) + \sec^2(x)\tan^9(x)\,dx \\ &= \int u^5 + 2u^7 + u^9\,du = \frac{1}{6}u^6 + \frac{1}{4}u^8 + \frac{1}{10}u^{10} + C \\ &= \frac{1}{6}\tan^6(x) + \frac{1}{4}\tan^8(x) + \frac{1}{10}\tan^{10}(x) + C.\end{aligned}$$

Alternatively, we could reduce until we have one tangent term, so we can set $u = \sec(x)$ and $du = \sec(x)\tan(x)\,dx$. We compute

$$\begin{aligned}\int \sec^6(x)\tan^5(x)\,dx &= \int \sec^6(x)\tan(x)(\sec^2(x)-1)^2\,dx \\ &= \int \sec^{10}(x)\tan(x) - 2\sec^8(x)\tan(x) + \sec^6(x)\tan(x)\,dx \\ &= \int u^9 - 2u^7 + u^5\,du = \frac{1}{10}u^{10} - \frac{1}{4}u^8 + \frac{1}{6}u^6 + C \\ &= \frac{1}{10}\sec^{10}(x) - \frac{1}{4}\sec^8(x) + \frac{1}{6}\sec^6(x) + C.\end{aligned}$$

These don't *look* the same, and they aren't—quite. They differ by $\frac{1}{60}$, but that's just a constant, so they are the same with the plus C. They're related by the identity that $\tan^2(x) + 1 = \sec^2(x)$, which is the identity we used to get these in the first place.

Problem 5 (Bonus)**.** Do one of the following two integrals. Explain why you don't want to do the other one.

(a) $\int \tan^2(x)\sec^3(x)\,dx$

(b) $\int \tan^3(x)\sec^3(x)\,dx$.

Solution: The second one is pretty straightforward. We can compute

$$\begin{aligned}\int \tan^3(x)\sec^3(x)\,dx &= \int \tan(x)\sec^5(x) - \tan(x)\sec^3(x)\,dx \\ &= \frac{1}{5}\sec^5(x) - \frac{1}{3}\sec^3(x) + C.\end{aligned}$$

The first one, on the other hand, is extremely painful. You can't get it to a form with a useful u substitution, and will have to use integration by parts and other painful work. The answer turns out to be

$$\int \tan^2(x)\sec^3(x)\,dx = \frac{1}{32}\Big(4\ln\left(\cos(x/2)-\sin(x/2)\right) - 4\ln\left(\cos(x/2)+\sin(x/2)\right) - \sec^4(x)\sin(3x) + 7\sec^3(x)\tan(x)\Big).$$

Problem 6. Consider the integral $\int \frac{dx}{\sqrt{4x^2-1}}$.

(a) Which trig function would let us simplify that square root, and what identity are we using?

(b) What trigonometric substitution should we use here?

(c) Compute the antiderivative.

(d) Make sure to substitute your x back into the equation!

Solution:

(a) We want to use a $\sec(\theta)$ form, because we can use the identity that $\sec^2(\theta)-1 = \tan^2(\theta)$.

(b) Set $2x = \sec(\theta)$, so that $x = \frac{1}{2}\sec(\theta)$. Then $dx = \frac{1}{2}\sec(\theta)\tan(\theta)\,d\theta$.

(c) We have

$$\begin{aligned}\int \frac{dx}{\sqrt{4x^2-1}} &= \int \frac{\frac{1}{2}\sec\theta\tan\theta\,d\theta}{\sqrt{\sec^2\theta-1}} \\ &= \frac{1}{2}\int \frac{\sec\theta\tan\theta\,d\theta}{\sqrt{\tan^2\theta}} \\ &= \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta+\tan\theta| + C.\end{aligned}$$

(d) We know that $\sec\theta = 2x$ by our definition of θ. To find $\tan\theta$ we draw a triangle: angle θ has hypotenuse $2x$ and adjacent side 1, and thus opposite side $\sqrt{4x^2-1}$, so $\tan\theta = \sqrt{4x^2-1}$. Thus

$$\int \frac{dx}{\sqrt{4x^2-1}}\,dx = \ln|x + \sqrt{4x^2-1}/2| + C.$$