Math 1232 Spring 2024
Single-Variable Calculus 2 Section 12
Mastery Quiz 5
Due Tuesday, February 20

This week's mastery quiz has three topics. Everyone should submit work on topic M2 (even if you got a 2 last week), and everyone should submit work on S3. If you have a 4/4 on M1, you don't need to submit them again. (Check Blackboard!)

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Calculus of Transcendental Functions
- Major Topic 2: Advanced Integration Techniques
- Secondary Topic 3: Numeric Integration

Name:

Recitation Section:

M1: Calculus of Invertible Functions

(a) Compute $\frac{d}{dx} x^{e^x}$

Solution:

$$\begin{aligned} y &= x^{e^x} \\ ln|y| &= e^x \ln|x| \\ y'/y &= e^x \ln|x| + \frac{e^x}{x} \\ y' &= e^x \ln|x| x^{e^x} + \frac{1}{x} e^x x^{e^x}. \end{aligned}$$

(b) $\int \frac{\cos(x)\sin(x)}{1+\cos^2(x)}\, dx =$

Solution: We can take $u = \cos(x)$ so that $du = -\sin(x)\, dx$. Then

$$\int \frac{\cos(x)\sin(x)}{1+\cos^2(x)}\, dx = \int \frac{-u}{1+u^2}\, du$$

Then we can set $v = 1 + u^2$ so that $dv = 2u\, du$ and we get

$$\begin{aligned} \int \frac{-u}{1+u^2}\, du &= \int \frac{-1}{2}\frac{1}{v}\, dv = \frac{-1}{2} \ln|v| + C \\ &= \frac{-1}{2} \ln|1+u^2| + C = \frac{-1}{2} \ln|1+\cos^2(x)| + C. \end{aligned}$$

(c) $\int \frac{x}{9+x^4}\, dx =$

Solution: We can factor a 9 out to get $\frac{1}{9}\frac{x}{1+x^4/9}$. Then we set $u = x^2/3$, and $du = 2x/3\, dx$, and we have

$$\begin{aligned} \int \frac{x}{9+x^4}\, dx &= \int \frac{1}{9}\frac{x}{1+u^2}\frac{3}{2x}\, du \\ &= \int \frac{1}{6}\frac{1}{1+u^2}\, du \\ &= \frac{1}{6} \arctan u + C = \frac{1}{6} \arctan(x^2/3) + C. \end{aligned}$$

M2: Advanced Integration Techniques

(a) Compute $\int \frac{x^2+x-4}{(x+3)^2(x+1)}\, dx =$

Solution:

$$\frac{x^2+x-4}{(x+3)^2(x+1)} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x+1}$$
$$x^2+x-4 = A(x+3)(x+1) + B(x+1) + C(x+3)^2$$
$$2 = -2B \Rightarrow B = -1$$
$$-4 = 4C \Rightarrow C = -1$$
$$-4 = 3A + B + 9C = 3A - 1 - 9 \Rightarrow A = 2$$
$$\frac{x^2+x-4}{(x+3)^2(x+1)} = \frac{2}{x+3} + \frac{-1}{(x+3)^2} + \frac{-1}{x+1}$$
$$\int \frac{x^2+x-4}{(x+3)^2(x+1)}\,dx = \int \frac{2}{x+3} + \frac{-1}{(x+3)^2} + \frac{-1}{x+1}\,dx$$
$$= 2\ln|x+3| + \frac{1}{x+3} - \ln|x+1| + C.$$

(b) Compute $\int \frac{x^2+x+3}{x-2}\,dx =$

Solution: Polynomial long division gives that

$$x^2+x+3 = x(x-2) + 3(x-2) + 9$$
$$\frac{x^2+x+3}{x-2} = x + 3 + \frac{9}{x-2}$$
$$\int \frac{x^2+x+3}{x-2}\,dx = \int x + 3 + \frac{9}{x-2}\,dx$$
$$= \frac{x^2}{2} + 3x + 9\ln|x-2| + C.$$

(c) $\displaystyle\int_0^{1/\sqrt{3}} \frac{x^3}{\sqrt{1+x^2}}\,dx =$

Solution: We'll set $x^2 = \tan^2(\theta)$ so $x = \tan(\theta)$, and then $dx = \sec^2(\theta)\,d\theta$. As x goes from 0 to $1/\sqrt{3}$ we have $\tan\theta$ goign from 0 to $1/\sqrt{3}$, and thus θ going from 0 to $\pi/6$. Then we have

$$\begin{aligned}
\int_0^{1/\sqrt{3}} \frac{x^3}{\sqrt{1+x^2}}\,dx &= \int_0^{\pi/6} \frac{\tan^3(\theta)}{\sqrt{1+\tan^2(\theta)}}\sec^2(\theta)\,d\theta \\
&= \int_0^{\pi/6} \tan^3(\theta)\sec(\theta)\,d\theta \\
&= \int_0^{\pi/6} \tan(\theta)\sec(\theta)(\sec^2(\theta)-1)\,d\theta \\
&= \frac{1}{3}\sec^3(\theta) - \sec(\theta)\Big|_0^{\pi/6} \\
&= \frac{8}{9\sqrt{3}} - \frac{2}{\sqrt{3}} - \left(\frac{1}{3}-1\right) = \frac{2}{3} - \frac{10}{9\sqrt{3}}.
\end{aligned}$$

S3: Numeric Integration

(a) How many intervals do you need with the **midpoint** rule to approximate $\int_2^4 \frac{1}{x}\,dx$ to within 1/100? Compute that approximation. (Feel free to use a calculator to plug in numeric values, or to leave the answer in exact unsimplified terms, but show every step.)

Solution: We have

$$\begin{aligned}
f'(x) &- 1/x^2 \\
f''(x) &= 2/x^3 \\
f''(2) &= 1/4 \\
|E_M| &\leq \frac{1/4 \cdot 2^3}{24 \cdot n^2} \leq \frac{1}{100} \\
n^2 &\geq 100/12 = 25/3 \approx 8.33 \\
n &\geq 4
\end{aligned}$$

so we need to use at least three intervals. Then the midpoint approximation would be

$$\begin{aligned}
\int_2^4 \frac{1}{x}\,dx =&\approx \frac{2}{3}\left(\frac{1}{7/3} + \frac{1}{9/3} + \frac{1}{11/3}\right) \\
&= \frac{2}{7} + \frac{2}{9} + \frac{2}{11} = \frac{478}{693} \approx .6898.
\end{aligned}$$

The true answer is approximately .6931 so we're well within 1/100.

If we use four intervals (which these original solutions did for some reason), then the midpoint approximation would be

$$\begin{aligned}
\int_2^4 \frac{1}{x}\,dx &\approx \frac{1}{2}\left(\frac{1}{2.25} + \frac{1}{2.75} + \frac{1}{3.25} + \frac{1}{3.75}\right) \\
&= \frac{2}{9} + \frac{2}{11} + \frac{2}{13} + \frac{2}{15} = \frac{4448}{6435} \approx .6912
\end{aligned}$$

The true answer is approximately .6931 so we're well within 1/100.

(b) Suppose we have

$$g(0) = 2.4 \qquad g(1) = 4 \qquad g(2) = 2.7 \qquad g(3) = 2.3 \qquad g(4) = 1.7$$

Approximate $\int_0^4 g(x)\,dx$ using the Trapezoid rule, and then using Simpson's rule.

Solution: For the trapezoid rule, we have

$$\begin{aligned} T_4 &= 1 \cdot \frac{2.4+4}{2} + 1 \cdot \frac{4+2.7}{2} + 1 \cdot \frac{2.7+2.3}{2} + 1 \cdot \frac{2.3+1.7}{2} \\ &= \frac{1}{2}(6.4+6.7+5+4.0) = \frac{1}{2} \cdot 22.1 = 11.05. \end{aligned}$$

For Simpson's rule, we have

$$\begin{aligned} S_4 &= \frac{1}{3}\left(2.4 + 4 \cdot 4 + 2 \cdot 2.7 + 4 \cdot 2.3 + 1.7\right) \\ &= \frac{1}{3}(2.4+12+5.4+9.2+1.9) = \frac{1}{3} \cdot 34.7 \approx 11.5667. \end{aligned}$$