Math 1232: Single-Variable Calculus 2
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Recitation 5

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Problem 1. We want to find $\int \frac{x^5+x-1}{x^3+1}\,dx$.

(a) What's the first tool we need to apply here? (Hint: not partial fractions!)

(b) Once we get it in a more manageable form, things should simplify out nicely. What is the final integral?

Solution:

(a) Because the numerator is higher degree than the denominator, we need to start with a polynomial long division. We get

$$\frac{x^5+x-1}{x^3+1} = x^2 - \frac{x^2-x+1}{x^3+1} = x^2 - \frac{x^2-x+1}{(x+1)(x^2-x+1)}.$$

(b) We don't even need to do a partial fractions decomposition here: instead it just factors. We have

Thus

$$\int \frac{x^5+x-1}{x^3+1}\,dx = \int x^2 - \frac{x^2-x+1}{(x+1)(x^2-x+1)}\,dx$$
$$= \int x^2 - \frac{1}{x+1}\,dx = \frac{x^3}{3} - \ln|x+1|.$$

Problem 2. We've looked briefly at the integral $\int \frac{1}{1+e^x}\,dx$. Let's try it again with our new tools.

(a) Try the substitution $u = e^x$. What do you get? What tools can apply to the result?

(b) Do a partial fractions decomposition to get the integral.

Solution:

(a) If $u = e^x$ then $du = e^x\,dx$.

$$\int \frac{1}{1+e^x}\,dx = \int \frac{1}{1+e^x}\frac{du}{e^x} = \int \frac{du}{u(1+u)}.$$

(b) This looks like a fraction with a factorable denominator. So a partial fractions decomposition gives us

$$\frac{1}{u(1+u)} = \frac{A}{u} + \frac{B}{1+u}$$
$$1 = A(1+u) + B(u) = A + (A+B)u$$

so $A = 1$ and $B = -1$.

(Alternatively, plugging in $u = 0$ gives $1 = A$, and plugging in $u = -1$ gives $1 = -B$.)

Thus

$$\begin{aligned}\int \frac{1}{1+e^x}\,dx &= \int \frac{du}{u(1+u)} \\ &= \int \frac{1}{u} - \frac{1}{1+u}\,du \\ &= \ln|u| - \ln|1+u| + C = \ln\left|\frac{e^x}{1+e^x}\right| + C.\end{aligned}$$

Problem 3 (Bonus)**.** Let's see if we can work out the integral of secant! This isn't at all obvious.

(a) We want $\int \sec(x)\,dx = \int \frac{1}{\cos(x)}\,dx$. Since this is a fraction, we can multiply the top and bottom through by $\cos(x)$. This makes the expression more complicated, but it does allow us to use a trig identity. What do we get?

(b) Now we can do a u substitution. What u substitution seems reasonable? Does it help us at all?

(c) Now we can use partial fractions to finish the problem off. We wind up with an awkward answer, but an answer.

(d) The most common formula for the integral of $\sec(x)$ is $\ln|\sec(x)+\tan(x)|+C$. Is that the same as what you got? (Hint: use logarithm laws and multiplication by the conjugate.)

Solution:

(a) We get $\int \frac{\cos(x)}{\cos^2(x)}\,dx$. The bottom allows us to use the pythagorean identity, and now we're trying to compute $\int \frac{\cos(x)}{1-\sin^2(x)}\,dx$.

(b) The only really reasonable choice here is $u=\sin(x)$, so $du=\cos(x)\,dx$. Then we have

$$\int \frac{\cos(x)}{1-\sin^2(x)}\,dx = \int \frac{1}{1-u^2}\,du.$$

(c) We write

$$\begin{aligned}\frac{1}{1-u^2} &= \frac{A}{1-u}+\frac{B}{1+u}\\ 1 &= A(1+u)+B(1-u).\end{aligned}$$

Plugging in $u=1$ gives us that $2B=1$ and plugging in $u=-1$ gives us $2A=1$, so we have

$$\begin{aligned}\int \frac{\cos(x)}{1-\sin^2(x)}\,dx &= \int \frac{1}{1-u^2}\,du\\ &= \frac{1}{2}\int \frac{1}{1-u}+\frac{1}{1+u}\,du\\ &= \frac{1}{2}\left(-\ln|1-u|+\ln|1+u|\right)+C\\ &= \frac{1}{2}\left(-\ln|1-\sin(x)|+\ln|1+\sin(x)|\right)+C.\end{aligned}$$

(d) It doesn't look the same, but it is!

$$\begin{aligned}\frac{1}{2}\left(-\ln|1-\sin(x)|+\ln|1+\sin(x)|\right) &= \frac{1}{2}\ln\left|\frac{1+\sin(x)}{1-\sin(x)}\right|\\ &= \frac{1}{2}\ln\left|\frac{(1+\sin(x))^2}{1-\sin^2(x)}\right|\\ &= \frac{1}{2}\ln\left|\frac{(1+\sin(x))^2}{\cos^2(x)}\right|\\ &= \ln\left|\frac{1+\sin(x)}{\cos(x)}\right|\\ &= \ln\left|\frac{1}{\cos(x)}+\frac{\sin(x)}{\cos(x)}\right| = \ln\left|\sec(x)+\tan(x)\right|.\end{aligned}$$

Problem 4 (Bonus)**.** What if we want to find $\int \frac{x^4+6x^3+4x^2+8x+11}{(x-1)^2(2x+1)(x^2+4x+5)}\,dx$?

Solution: The numerator is lower degree than the denominator, so we can begin a partial fraction decomposition immediately. We set up:

$$\begin{aligned}
\frac{x^4+6x^3+4x^2+8x+11}{(x-1)^2(2x+1)(x^2+4x+5)} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{2x+1} + \frac{Dx+E}{x^2+4x+5} \\
x^4+6x^3+4x^2+8x+11 &= A(x-1)(2x+1)(x^2+4x+5) + B(2x+1)(x^2+4x+5) \\
&\quad + C(x-1)^2(x^2+4x+5) + (Dx+E)(x-1)^2(2x+1) \\
&= A(2x^4+7x^3+5x^2-9x-5) + B(2x^3+9x^2+14x+5) \\
&\quad + C(x^4+2x^3-2x^2-6x+5) + D(2x^4-3x^3+x) \\
&\quad + E(2x^3-3x^2+1) \\
&= (2A+C+2D)x^4 + (7A+2B+2C-3D+2E)x^3 \\
&\quad + (5A+9B-2C-3E)x^2 + (-9A+14B-6C+D)x \\
&\quad + (-5A+5B+5C+E)
\end{aligned}$$

We get a system of equations

$$\begin{aligned}
2A+C+2D &= 1 & 7A+2B+2C-3D+2E &= 6 \\
5A+9B-2C-3E &= 4 & -9A+14B-6C+D &= 8 \\
-5A+5B+5C+E &= 11.
\end{aligned}$$

After solving this (admittedly nasty) collection of equations, we see that $A=0, B=1, C=1, D=0, E=1$. So

$$\begin{aligned}
\int \frac{x^4+6x^3+4x^2+8x+11}{(x-1)^2(2x+1)(x^2+4x+5)}\,dx &= \int \frac{1}{(x+1)^2} + \frac{1}{2x+1} + \frac{1}{x^2+4x+5}\,dx \\
&= -(x+1)^{-1} + \frac{1}{2}\ln|2x+1| + \int \frac{dx}{x^2+4x+5}.
\end{aligned}$$

We complete the square, and see that $x^2+4x+5 = (x+2)^2+1$, so we use $u = x+2$ and get

$$\int \frac{dx}{x^2+4x+5} = \int \frac{du}{u^2+1} = \arctan(u) + C = \arctan(x+2) + C.$$

Thus

$$\int \frac{x^4+6x^3+4x^2+8x+11}{(x-1)^2(2x+1)(x^2+4x+5)}\,dx = -(x+1)^{-1} + \frac{1}{2}\ln|2x+1| + \arctan(x+2) + C.$$

Problem 5. We want to find the cross-sectional area of a two-meter-long airplane wing. We measure its width every 20 centimeters, and get: 5.8, 20.3, 26.7, 29.0, 27.6, 27.3, 23.8, 20.5, 15.1, 8.7, 2.8. Use the trapezoidal rule and Simpson's rule to estimate the area of the wing.

Solution:

$$\begin{aligned}
T_{10} &= .2\left(\frac{5.8+20.3}{2}+\frac{20.3+26.7}{2}+\frac{26.7+29.0}{2}+\frac{29.0+27.6}{2}+\frac{27.6+27.3}{2}\right.\\
&\quad\left.+\frac{27.3+23.8}{2}+\frac{23.8+20.5}{2}+\frac{20.5+15.1}{2}+\frac{15.1+8.7}{2}+\frac{8.7+2.8}{2}\right)\\
&= 40.66\\
S_{10} &= \frac{1}{15}(5.8+4\cdot 20.3+2\cdot 26.7+4\cdot 29.0+2\cdot 27.6+4\cdot 27.3\\
&\quad +2\cdot 23.8+4\cdot 20.5+2\cdot 15.1+4\cdot 8.7+2.8)\\
&= \frac{618.2}{15} = 41.2133.
\end{aligned}$$

Problem 6. Consider the function $f(x) = x^2 + 1$.

(a) Use the trapezoid rule with six intervals to estimate $\int_{-4}^{2} f(x)\,dx$.

(b) Use the midpoint rule with six intervals to estimate $\int_{-4}^{2} f(x)\,dx$.

(c) Use Simpson's rule with six intervals to estimate $\int_{-4}^{2} f(x)\,dx$.

(d) Which of these do you expect to be most accurate? Which do you expect to be least accurate?

(e) Compute $\int_{-4}^{2} f(x)\,dx$. What do you find? Why?

Solution:

(a)

$$\begin{aligned}
\int_{-4}^{2} x^2+1\,dx &\approx \frac{f(-4)+f(-3)}{2}+\frac{f(-3)+f(-2)}{2}+\frac{f(-2)+f(-1)}{2}\\
&\quad+\frac{f(-1)+f(0)}{2}+\frac{f(0)+f(1)}{2}+\frac{f(1)+f(2)}{2}\\
&= \frac{17+10}{2}+\frac{10+5}{2}+\frac{5+2}{2}+\frac{2+1}{2}+\frac{1+2}{2}+\frac{2+5}{2}\\
&= \frac{1}{2}(27+15+7+3+3+7) = \frac{62}{2} = 31.
\end{aligned}$$

Alternatively, we could write

$$\begin{aligned}
\int_{-4}^{2} x^2+1\,dx &\approx \frac{1}{2}\Big(f(-4)+2f(-3)+2f(-2)+2f(-1)+2f(0)+2f(1)+f(2)\Big)\\
&= \frac{1}{2}\Big(17+20+10+4+2+4+5\Big) = \frac{1}{2}\cdot 62 = 31.
\end{aligned}$$

(b)

$$\int_{-4}^{2} x^2 + 1\,dx \approx f(-3.5) + f(-2.5) + f(-1.5) + f(-0.5) + f(0.5) + f(1.5)$$
$$= 13.25 + 7.25 + 3.25 + 1.25 + 1.25 + 3.25 = 29.5.$$

(c)

$$\int_{-4}^{2} x^2 + 1\,dx \approx \frac{1}{3}\Big(f(-4) + 4f(-3) + 2f(-2) + 4f(-1) + 2f(0) + 4f(1) + f(2)\Big)$$
$$= \frac{1}{3}\Big(17 + 40 + 10 + 8 + 2 + 8 + 5\Big) = 30.$$

(d) We'd expect these to be roughly in increasing order of accuracy, with the trapezoid rule being the least accurate, and the midpoint rule being the most accurate.

(e)

$$\begin{aligned}\int_{-4}^{2} x^2 + 1\,dx &= \frac{x^3}{3} + x\Big|_{-4}^{2} \\ &= \frac{8}{3} + 2 - \left(\frac{-64}{3} - 4\right) \\ &= \frac{72}{3} + 6 = 24 + 6 = 30.\end{aligned}$$

Simpson's rule got it exactly correct! This makes sense, because the fourth derivative is zero.

Problem 7. Let $g(x) = e^{-x^2}$, and suppose we want to compute $\int_{-1}^{2} e^{-x^2}\,dx$, and get the answer correct to two decimal places.

(a) We can compute that $g''(x)$ varies between -2 and $.9$ when x is in $[-1, 2]$. What value should we take for K?

(b) How many subintervals should we use to get the answer correct to within two decimal places using the trapezoid rule?

(c) How many subintervals should we use to get the answer correct to within two decimal places using the midpoint rule?

(d) We can compute that $g''''(x)$ varies between -8 and 12. What value should we take for L?

(e) How many subintervals should we use to get the answer correct to within two decimal places using Simpson's rule?

Solution:

(a) Since $g''(x)$ varies between -2 and $.9$, the largest the absolute value can get will be 2. Thus we should take $K = 2$.

If we wanted to compute this for ourselves, we could observe that

$$\begin{aligned} g''(x) &= 2e^{-x^2}(2x^2 - 1) \\ g'''(x) &= -4xe^{-x^2}(2x^2 - 3). \end{aligned}$$

We want to maximize g'' so we look for the zeroes of g''', which happen at 0 and at $\pm\sqrt{3/2}$. We can ignore $-\sqrt{3/2}$ since it's not in $[-1, 2]$, but we do need to check the endpoints, so we compute

$$\begin{aligned} g''(-1) &\approx .74 & g''(0) &= -2 \\ g''(\sqrt{3/2}) &\approx .89 & g''(2) &\approx .26. \end{aligned}$$

Thus the absolute minimum is -2 and the absolute max is about $.89$.

(b) The error in the trapezoid rule is

$$\begin{aligned} |E_T| \leq \frac{2(3)^3}{12n^2} &< \frac{1}{100} \\ 5400 &< 12n^2 \\ 450 &< n^2 \\ 21.2 &< n \end{aligned}$$

so we'd need at least 22 intervals.

(c) The error in the midpoint rule is

$$\begin{aligned} |E_T| \leq \frac{2(3)^3}{24n^2} &< \frac{1}{100} \\ 5400 &< 24n^2 \\ 225 &< n^2 \\ 15 &< n \end{aligned}$$

so we'd need *sixteen* intervals to make sure the error was *less than* 1/100, but 15 will guarantee the error is *at most* 1/100.

(d) We need to take $L = 12$.

(e) The error in Simpson's rule is

$$\begin{aligned} |E_S| \leq \frac{12 \cdot 3^5}{180n^4} &< \frac{1}{100} \\ 291600 &< 180n^4 \\ 1620 &< n^4 \\ 6.344 &< n^4 \end{aligned}$$

so we'd only need five intervals.