Math 1232 Spring 2024
Single-Variable Calculus 2 Section 12
Mastery Quiz 6
Due Tuesday, February 27

This week's mastery quiz has four topics. Everyone should submit work on topic S4 and S5. If you have a 4/4 on M2, or a 2/2 on S3, you don't need to submit them again. (Check Blackboard!)

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 2: Advanced Integration Techniques
- Secondary Topic 3: Numeric Integration
- Secondary Topic 4: Improper Integrals
- Secondary Topic 5: Geometric Applications

Name:

Recitation Section:

M2: Advanced Integration Techniques

(a) $\displaystyle\int \frac{\sqrt{4x^2-1}}{x}\,dx =$

Solution: We set $2x = \sec(\theta)$, so $dx = \frac{1}{2}\sec(\theta)\tan(\theta)\,d\theta$, and

$$\begin{aligned}
\int \frac{\sqrt{4x^2-1}}{x}\,dx &= \int \frac{\sqrt{\sec^2\theta - 1}}{\frac{1}{2}\sec\theta}\frac{1}{2}\sec(\theta)\tan\theta\,d\theta \\
&= \int \tan^2(\theta)\,d\theta = \int \sec^2(\theta) - 1\,d\theta \\
&= \tan(\theta) - \theta + C
\end{aligned}$$

Then we know $\sec\theta = 2x$ so we can make a triangle with hypotenuse $2x$ and adjacent side 1, and thus opposite side $\sqrt{4x^2-1}$, so $\tan(\theta) = \sqrt{4x^2-1}$. Then we can say either $\theta = \operatorname{arcsec}(2x)$ or $\theta = \arctan(\sqrt{4x^2-1})$, and we have

$$\int \frac{\sqrt{4x^2-1}}{x}\,dx = \sqrt{4x^2-1} - \arctan(\sqrt{4x^2-1}) + C = \sqrt{4x^2-1} - \operatorname{arcsec}(2x) + C.$$

(b) $\displaystyle\int_0^3 x^2e^{2x}\,dx =$

Solution:

$$\begin{aligned}
\int_0^3 x^2e^{2x}\,dx &= \frac{1}{2}x^2e^{2x}\Big|_0^3 - \int_0^3 xe^{2x}\,dx \\
&= \frac{9}{2}e^6 - \int_0^3 xe^{2x}\,dx \\
\int_0^3 xe^{2x}\,dx &= \frac{1}{2}xe^{2x}\Big|_0^3 - \int_0^3 \frac{1}{2}e^{2x}\,dx \\
&= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\Big|_0^3 \\
&= \frac{3}{2}e^6 - \frac{1}{4}e^6 + \frac{1}{4} \\
\int_0^3 x^2e^{2x}\,dx &= \frac{9}{2}e^6 - \frac{3}{2}e^6 + \frac{1}{4}e^6 - \frac{1}{4} = \frac{13}{4}e^6 - \frac{1}{4}.
\end{aligned}$$

(c) Compute $\displaystyle\int \frac{4x^2-x+10}{(x-2)(x^2+4)}\,dx$.

Solution: We need to do a partial fractions decomposition. We have

$$\begin{aligned}
\frac{4x^2-x+10}{(x-2)(x^2+4)} &= \frac{A}{x-2}+\frac{Bx+C}{x^2+4} \\
4x^2-x+10 &= A(x^2+4)+(Bx+C)(x-2).2: \quad 24 &= A\cdot 8 \\
A &= 30: \quad 10 &= 4A+(C)(-2) = 12-2C \\
C &= 6-5=1 \\
1: \quad 13 &= 3(1+4)+(B+1)(1-2) = 15-B-1 \\
B &= 1.
\end{aligned}$$

Thus we compute

$$\begin{aligned}
\int \frac{4x^2-x+10}{(x-2)(x^2+4)}\,dx &= \int \frac{3}{x-2}+\frac{x+1}{x^2+4}\,dx \\
&= \int \frac{3}{x-2}+\frac{x}{x^2+4}+\frac{1}{x^2+4}\,dx \\
&= 3\ln|x-2|+\frac{1}{2}\ln|x^2+4|+\frac{1}{4}\int\frac{1}{(x/2)^2+1}\,dx \\
&= 3\ln|x-2|+\frac{1}{2}\ln|x^2+4|+\frac{1}{2}\arctan(x/2)+C.
\end{aligned}$$

S3: Numeric Integration

(a) How many intervals do you need with the **trapezoid** rule to approximate $\int_5^9 (x+4)^{3/2}\,dx$ to within $1/10$? Use the trapezoid rule with that many intervals to approximate the integral.

(Feel free to use a calculator to plug in numeric values, or to leave the answer in exact unsimplified terms, but show every step.)

Solution: We have

$$\begin{aligned}
f'(x)\frac{3}{2}(x+4)^{1/2}f''(x) &= \frac{3}{4}(x+4)^{-1/2} = \frac{3}{4\sqrt{x+4}} \\
f''(5) &= \frac{1}{4} \\
|E_M| &\leq \frac{{}^1\!/_4\cdot 4^3}{12\cdot n^2} \leq \frac{1}{10} \\
n^2 &\geq 40/3 \approx 13.3 \\
n &\geq 4
\end{aligned}$$

so we need to use at least four intervals. Then the midpoint approximation would be

$$\begin{aligned}
\int_5^9 (x+4)^{3/2}\,dx &\approx \frac{\sqrt{9}^3+\sqrt{10}^3}{2}+\frac{\sqrt{10}^3+\sqrt{11}^3}{2}+\frac{\sqrt{11}^3+\sqrt{12}^3}{2}+\frac{\sqrt{12}^3+\sqrt{13}^3}{2} \\
&\approx \frac{1}{2}9^{3/2}+10^{3/2}+11^{3/2}+12^{3/2}+\frac{1}{2}13^{3/2}.
\end{aligned}$$

We can stop there, but numerically this is roughly 146.61. The true answer is approximately 146.54 so this is within the expected error bound.

(b) Suppose we have

$$g(0) = 5 \quad g(1) = 4 \quad g(2) = 2 \quad g(3) = 3 \quad g(4) = 5 \quad g(5) = 6 \quad g(6) = 5$$

Approximate $\int_0^6 g(x)\,dx$ using the midpoint rule and the Simpson's rule.

Solution: For the midpoint rule, we have

$$T_3 = 2g(1) + 2g(3) + 2g(5) = 8 + 6 + 12 = 26.$$

For the Simpson's rule, we have

$$S_6 = \frac{1}{3}(5 + 16 + 4 + 12 + 10 + 24 + 5) = \frac{76}{3} \approx 25.33.$$

S4: Improper Integrals

(a) Compute $\displaystyle\int_1^2 \frac{dx}{x\ln(x)} =$

Solution:

$$\begin{aligned}\int_1^2 \frac{dx}{x\ln(x)} &= \lim_{t\to1^+}\int_t^2 \frac{dx}{x\ln(x)}\\ &= \lim_{t\to1^+} \ln(|\ln(x)|)\Big|_t^2\\ &= \lim_{t\to1^+} \ln(|\ln(2)|) - \ln|\ln(t)|\end{aligned}$$

But $\lim_{t\to1^+}\ln(t) = 0$, so $\lim_{t\to1^+}\ln|\ln(t)| = -\infty$. So this limit diverges.

(b) Compute $\displaystyle\int_1^\infty xe^{-x^2}\,dx.$

Solution: We'll take $u = -x^2$ so $du = -2x\,dx$. Then

$$\begin{aligned}\int_1^\infty xe^{-x^2}\,dx &= \lim_{t\to\infty}\int_1^t xe^{-x^2}\,dx \\ &= \lim_{t\to\infty}\int_{-1}^{-t^2}\frac{-1}{2}e^u\,du \\ &= \lim_{t\to\infty}\frac{-1}{2}e^u\Big|_{-1}^{-t^2} \\ &= \frac{-1}{2}\lim_{t\to\infty} e^{-t^2}-e^{-1} \\ &= \frac{-1}{2}(0-e^{-1}) = \frac{1}{2e}.\end{aligned}$$

S5: Geometric Applications

(a) Compute the arc length of the curve $y = \frac{x^4}{8} + \frac{1}{4x^2}$ as x varies from 2 to 4.

Solution: We have $y' = \frac{x^3}{2} + \frac{-1}{2x^3}$, and thus

$$\begin{aligned}L &= \int_2^4 \sqrt{1+(x^3/2-x^{-3}/2)^2}\,dx = \int_2^4 \sqrt{1+x^6/4-1/2+x^{-6}/4}\,dx \\ &= \int_2^4 \sqrt{x^6/4+1/2+x^{-6}/4}\,dx = \int_2^4 \sqrt{(x^3/2+x^{-3}/2)^2}\,dx \\ &= \int_2^4 x^3/2+x^{-3}/2\,dx = \frac{x^4}{8}-\frac{1}{4x^2}\Big|_2^4 \\ &= \frac{256}{8}-\frac{1}{64}-\left(\frac{16}{8}-\frac{1}{16}\right) = \frac{1923}{64}.\end{aligned}$$

(b) Set up (but don't compute!) an integral for the area of a surface obtained by taking the curve $y = \ln(x^3+1)$ from $x = 0$ to $x = 10$ and rotating around the x axis.

Solution: We have $y' = \frac{3x^2}{x^3+1}$ so we get

$$\begin{aligned}SA &= \int_0^{10} 2\pi y\sqrt{1+y'^2}\,dx \\ &= \int_0^{10} 2\pi\ln(x^3+1)\sqrt{1+\frac{9x^4}{(x^3+1)^2}}\,dx.\end{aligned}$$