Math 1232: Single-Variable Calculus 2
George Washington University Spring 2023
Recitation 6

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Problem 1. We want to compute $\int_0^2 \frac{1}{\sqrt[3]{x-1}}\, dx$.

(a) Can you compute an antiderivative? Can you evaluate it at 0 and 2?

(b) Did part (a) finish the problem? Sketch a picture of the graph. What should we be concerned about?

(c) Carefully set up a computation that will find $\int_0^2 \frac{1}{\sqrt[3]{x-1}}\, dx$. (Hint: It should have two limit operations in it.)

(d) What did we learn from this that we didn't learn from (a)?

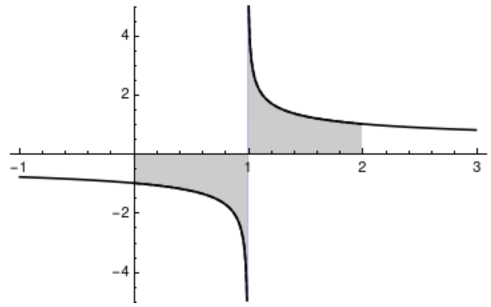
Solution:

(a) We know that

$$\int \frac{1}{\sqrt[3]{x-1}}\, dx = \frac{3}{2}(x-1)^{2/3}$$
$$\frac{3}{2}(x-1)^{2/3}\Big|_0^2 = \frac{3}{2}(1)^{2/3} - \frac{3}{2}(-1)^{2/3} = 0.$$

(b) This is not a correct way to compute this integral, because we're assuming everything makes sense in the middle. But it does not.

This is an improper integral, because the function is discontinuous and undefined at 1.



(c) To compute, we break it apart and compute a limit:

$$\begin{aligned}
\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx &= \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^2 \frac{1}{\sqrt[3]{x-1}}\,dx \\
&= \lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt[3]{x-1}}\,dx + \lim_{s\to 1^+} \int_s^2 \frac{1}{\sqrt[3]{x-1}}\,dx \\
&= \lim_{t\to 1^-} \left(\frac{3}{2}(x-1)^{2/3}\right)\Big|_0^t - \lim_{s\to 1^+} \left(\frac{3}{2}(x-1)^{2/3}\right)\Big|_s^2 \\
&= \lim_{t\to 1^-} \left(\frac{3}{2}(t-1)^{2/3} - \frac{3}{2}\cdot 1\right) + \lim_{s\to 1^+} \left(\frac{3}{2}\cdot 1 - \frac{3}{2}(s-1)^{2/3}\right) \\
&= 0 - 3/2 + 3/2 - 0 = 0.
\end{aligned}$$

(d) We get the "same answer" here, of 0. But this computation gives us an extra piece of information: all the area involved is finite. Saying the net area is $3/2 - 3/2$ is very different from trying to say it's $\infty - \infty$ somehow.

Problem 2. We want to figure out if $\int_0^{+\infty} e^{-x^2}\,dx$ converges—that is, if it's finite or infinite.

(a) If we can find an antiderivative, we can just compute the improper integral directly. Why doesn't that work?

(b) Since we can't integrate this directly we might want to use a comparison test. We need to find an easy-to-integrate function that's larger than e^{-x^2}. Find a function $f(x)$ that makes $f(x)e^{-x^2}$ easy to integrate.

(c) If $f(x) \geq 1$, then we can just integrate $f(x)e^{-x^2}$. Is it?

(d) This is where we can pull in a trick. Is there some a where $f(x) > 1$ when $x > a$? (You may need to adjust your $f(x)$ here, especially the sign. It's fine as long as you can still integrate it.)

(e) We know $\int_a^{+\infty} e^{-x^2}\,dx \leq \int_a^{+\infty} f(x)e^{-x^2}\,dx$. Compute the new improper integral; is it finite?

(f) Now we just have to deal with $\int_0^a e^{-x^2}\,dx$. We can't do that integral exactly, but that's fine: you should be able to tell whether it's finite or not without doing any calculations. How?

(g) Does $\int_0^{+\infty} e^{-x^2}\,dx$ converge?

Solution:

(a) This is precisely the standard function we know we don't have an elementary antiderivative for. So that won't help.

(b) The obvious value for $f(x)$ is $-2x$, since that's the chain rule from e^{-x^2}. It will work out easier in the long run if we take $f(x) = 2x$, though.

(c) Clearly $-2xe^{-x^2}$ is often less than e^{-x^2}, since the first is negative and the second is positive. Even if we fix that problem, we still see that $2xe^{-x^2} < e^{-x^2}$ when $x < 1/2$.

(d) If we take $a = 1/2$, or any larger number, we fix this problem. I'm going to take $a = 1$; then we have $2xe^{-x^2} > e^{-x^2}$ for $x > 1$.

(e) In class we showed that $\int_1^{+\infty} 2xe^{-x^2}\,dx$ converges. In particular we showed that

$$\begin{aligned}\int_1^{+\infty} 2xe^{-x^2}\,dx &= \lim_{s\to+\infty}\int_1^s 2xe^{-x^2}\,dx\\ &= \lim_{s\to+\infty}\int_1^{s^2} e^{-u}\,du\\ &= \lim_{s\to+\infty} -e^{-u}\Big|_1^{s^2}\\ &= \lim_{s\to+\infty} -e^{-s^2} - (-e^{-1}) = 0 + \frac{1}{e} = \frac{1}{e}.\end{aligned}$$

So we know that

$$\int_1^{+\infty} e^{-x^2}\,dx < \int_1^{+\infty} 2xe^{-x^2}\,dx = \frac{1}{e}$$

and thus $\int_1^{+\infty} e^{-x^2}\,dx$ converges.

(f) $\int_0^1 e^{-x^2}\,dx$ isn't an integral we can do. But it's a nice, proper integral, so nothing weird can happen. An integral of a function that's defined and continuous everywhere on the closed interval $[0, 1]$ will always *converge*. (Numerically, it works out to about 0.75.)

(g) We have concluded that

$$\int_0^{+\infty} e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^{+\infty} e^{-x^2}\,dx$$
$$< \int_0^1 e^{-x^2}\,dx + \frac{1}{e} < \infty.$$

In fact, this tells us that $\int_0^{+\infty} e^{-x^2}\,dx < \frac{1}{e} + .75 \approx 1.12$.

Using fancy techniques from cmoplex analysis, we can determine that $\int_0^{+\infty} e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi} \approx .88$. That's way outside the scope of what we can do in this course, though.

Problem 3. Let $f(x) = x^2$. Let's find the arc length between $x = 0$ and $x = 4$.

(a) This makes a very reasonable shape. What does the graph look like?

(b) Set up an integral to compute this arc length. You need to think about the variable of integration, the bounds, and the actual function to integrate.

(c) What techniques should we use to compute this integral? Where do we get stuck?

(d) Is there another way we could have set it up?

(e) Is that integral any easier?

Solution:

(a) It's a parabola. We know parabolas.

(b) The problem setup tells us we have x going from 0 to 4. And $f'(x) = 2x$. So it makes sense to set up

$$L = \int_0^4 \sqrt{1 + (2x)^2}\,dx$$

(c) This looks a lot like a trig sub integral. We can set $2x = \tan\theta$, so $dx = \frac{1}{2}\sec^2\theta\,d\theta$. When $x = 0$ we have $\tan\theta = 0$ so $\theta = 0$, and when $x = 4$ we have $\tan\theta = 8$ so $\theta = \arctan(8)$. This gives us

$$L = \int_0^4 \frac{1}{2}\sqrt{1 + (2x)^2}\,dx = \int_0^{\arctan 8} \frac{1}{2}\sqrt{1 + \tan^2(\theta)}\sec^2\theta\,d\theta$$
$$= \int_0^{\arctan 8} \frac{1}{2}\sec^3\theta\,d\theta$$

and at this point I...give up on this integral. We said that functions with odd numbers of secants and even numbers of tangents suck, and we just don't want to do them. But we can look it up, or plug it into a computer algebra package. We get

$$\frac{1}{4}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)|\Big|_0^{\arctan 8} \approx 16.819.$$

(d) We could instead have treated the function as $x = \sqrt{y}$, with $\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$. Then we get

$$L = \int_0^{16} \sqrt{1 + \frac{1}{4y}}\, dy.$$

(e) This integral is also technically doable, but very unpleasant and it involves both integration by parts and hyperbolic trig functions.

Problem 4. Let $f(x) = \sqrt[3]{3x}$. Take the portion of the graph where $0 \leq y \leq 2$ and rotate it around the y axis.

(a) Try to sketch a picture of what this will look like.

(b) Set up an integral to find the surface area. Again, think about the variable of integration, the bounds, and the function. Do you have multiple choices here or just one?

(c) Can you compute that integral?

Solution:

(a)

(b) Because we're rotating around the y axis, we basically have to integrate with respect to y. So we have $y = \sqrt[3]{3x}$ and thus $x = y^3/3$. Then $x' = y^2$, and we have

$$SA = \int_0^2 \frac{2\pi y^3}{3}\sqrt{1+y^4}\, dy.$$

(c) This one isn't so bad!

$$\begin{aligned} SA &= \int_0^2 \frac{2\pi y^3}{3}\sqrt{1+y^4}\, dy \\ &= \frac{2\pi}{3}\int_0^2 y^3\sqrt{1+y^4}\, dy \\ &= \frac{2\pi}{3}\frac{2}{12}(1+y^4)^{3/2}\Big|_0^2 \\ &= \frac{\pi}{9}\left(17^{3/2}-1\right) \approx 24.118. \end{aligned}$$

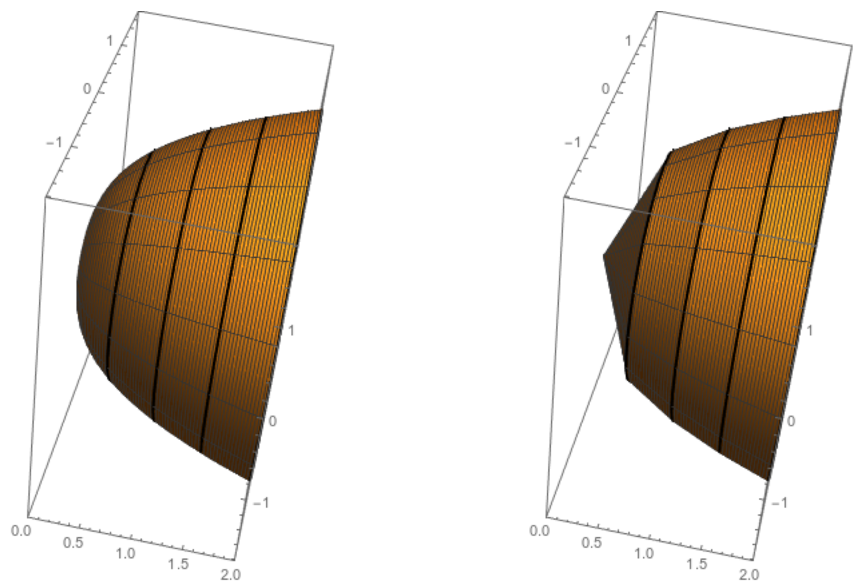


Figure 0.1: The graph of $y = \sqrt[3]{3x}$ rotated around the x-axis