Math 1232: Single-Variable Calculus 2
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Recitation 7

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Problem 1. Suppose we have a Hooke's Law system of a weight on a spring. Suppose $m = k$, so that we get the differential equation $x''(t) = -x(t)$.

(a) From class, we know the general solution to this differential equation. What is it?

(b) Suppose now we start (at time 0) with the weight stationary and displaced by 1 meter. What initial conditions does this correspond to?

(c) Find the specific solution to this initial value problem.

(d) What does this describe physically? Does that solution make physical sense?

Solution:

(a) We said that then $x(t) = a\sin(t) + b\cos(t)$ for some constants a and b.

(b) These are the position and velocity at time zero, so we have $x(0) = 1$ and $x'(0) = 0$.

(c) Since $x(0) = 1$ we know that

$$1 = a\sin(0) + b\cos(0) = b,$$

and since $x'(0) = 0$ we know that

$$0 = a\cos(0) - b\sin(0) = a$$

so we have $a = 0, b = 1$, and $x(t) = \cos(t)$ is the specific solution.

(d) The weight is oscillating between plus one meter and minus one meter. This makes sense: it starts at one meter of displacement, with no velocity, so that's as far as it should ever get.

Problem 2. (a) Can you find a (non-trivial) solution to $x''(t) = -2x(t)$? Make sure to check that your solution is right.

(b) Can you find a (non-trivial) solution to $x''(t) = 2x(t)$? Make sure to check that your solution is right.

Solution:

(a) We need to throw in a constant. The obvious solutions are $\sin(\sqrt{2}t)$ or $\cos(\sqrt{2}t)$; the general form of the solution is $A\sin(\sqrt{2}t) + B\cos(\sqrt{2}t)$.

(b) Here the "obvious" solutions are $e^{\sqrt{2}t}$ and $e^{-\sqrt{2}t}$. The general solution is $Ae^{\sqrt{2}t}+Be^{-\sqrt{2}t}$.

These solutions are similar; this is another example of our principle that trig functions and exponential functions are basically the same thing.

As a bonus: can you find a solution to $x''(t) = -2x(t)$ of the form e^{kt}? Hint: think about complex numbers.

Problem 3. Find the general solution to $(y^2 + xy^2)y' = 1$.

Solution: We have

$$\begin{aligned}
\int y^2 dy &= \int \frac{dx}{1+x} \\
\frac{y^3}{3} &= \ln|1+x| + C \\
y &= \sqrt[3]{3\ln|1+x| + 3C}.
\end{aligned}$$

Problem 4 (Evans Price Change Model)**.** In this problem we want to create an economic model of how prices change over time. Let's imagine we're studying a market where people buy and sell pairs of headphones.

(a) First we need to understand supply and demand curves. We can write a demand function $D(p)$ that takes in a price p and tells us the number of people who would want to buy a pair of headphones at that price on a given day. Should this function have a positive or negative derivative, and why?

(b) We can also write a supply function $S(p)$ that takes in a price p, and tells us the number of pairs of headphones that suppliers are willing to sell at that price each day. Should this function have a positive or negative derivative, and why?

(c) Suppose we have $D(p) = 80 - 5p$ and $S(p) = 5p - 20$. What will be the equilibrium price, where the quantity demanded (number of headphones people want to buy) is the same as the quantity supplied. How many headphones will be sold each day?

(d) That gives us the equilibrium price. But people actually trading in the market don't know the equilibrium price; it takes actual time to find it. We want to study the function $p(t)$, which tells us the price as a function of time.

The Evans model of price change says that the price changes at a rate proportional to the difference between the quantity demanded and the quantity supplied. That is, if a lot more people want to buy than sell, the price will rise quickly; if only a few people more people want to buy than sell, the price will still rise, but slowly.

Spend a couple minutes trying to write down a differential equation that encodes this model. Talk to your neighbors, see if they agree!

(e) If you didn't already, make sure your equation only has p as a variable. But there should also be a constant k. What does k represent?

(f) Let's assume $k = 1$. Do you recognize this equation? Can you find the general form of the solution?

(g) Does the trivial solution $p(t) = 0$ solve this system? Does that make sense?

(h) What is the limit as $t \to +\infty$? Does that make sense?

(i) Can you find a specific solution when $p(0) = 15$?

Solution:

(a) The derivative should be negative; fewer people will buy at a higher price.

(b) This derivative should be positive; more people will sell at a higher price.

(c) We want

$$\begin{aligned} D(p) &= S(p) \\ 80 - 5p &= 5p - 20 \\ 200 &= 10p \\ 10 &= p \end{aligned}$$

so these headphones will sell for \$10 each in equilibrium. At this price, thirty pairs of headphones will be sold each day.

(d) At the most schematic we can say $p' = k(D-S)$: the rate of price change is proportional to the difference between demand and supply.

In this case we know $D = 80 - 5p$ and $S = 5p - 20$, so we get

$$p' = k(100 - 10p).$$

(e) The k is a constant of proportionality. It represents how quickly prices adjust: if k is large then the price will change quickly, but if k is small it takes them a long time to change.

(f) We now have the equation $p' = 100 - 10p$. This is the mixing equation we studied! It a separable equation, and we can write

$$\begin{aligned} \frac{dp}{dt} &= 100 - 10p \\ \frac{dp}{100 - 10p} &= dt \\ \int \frac{dp}{100 - 10p} &= \int dt \\ \frac{-1}{10} \ln|100 - 10p| &= t + C \\ \ln|100 - 10p| &= -10t - 10C \\ 100 - 10p &= e^{-10t-10C} = C_1 e^{-10t} \\ 10p &= 100 - C_1 e^{-10t} \\ p &= 10 - \frac{C_1}{10} e^{-10t} = 10 - C_2 e^{-10t}. \end{aligned}$$

(g) The trivial solution doesn't work here; we can tell that either because there's a constant term of 10 in the general form of the solution, or because if $p = 0$ constantly then $\frac{dp}{dt} = 0$ and we get $0 = 100 - 0$ which is not true.

(h) $\lim_{t \to +\infty} p = 10$. This should make sense, because the equilibrium price is \$10.

(i) If $p(0) = 15$, we have

$$15 = 10 - C_2 e^{-10 \cdot 0} = 10 - C_2$$

so $C_2 = -5$ and the specific solution is

$$p = 10 + 5e^{-10t}.$$