Math 1232: Single-Variable Calculus 2
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Recitation 8

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Problem 1. Let $(a_n) = \left(-6, 4, \frac{-8}{3}, \frac{16}{9}, \frac{-32}{27}, \ldots\right)$.

(a) Find a closed-form formula for a_n.

(b) Is there a real function f so that $f(n) = a_n$?

(c) What is $\lim_{n\to\infty} a_n$? Why?

Solution:

(a) $a_n = 6 \cdot \left(\frac{-2}{3}\right)^n$.

(b) There isn't really a natural one, because you can't just take $\left(\frac{-2}{3}\right)^x$ for x irrational. (Or for x rational with even denominator; you can't take the square root.)

It is *possible* to find a function that interpolates this, though. It's just adding a bunch of noise. A good example would be

$$f(x) = 6 \cdot \left(\frac{2}{3}\right)^n \cos(n\pi).$$

(c) The limit is zero. There are a few ways to argue this, but they pretty much all fall back to the squeeze theorem.

My approach would be to observe that

$$-6 \cdot \frac{2^n}{3^n} \leq a_n \leq 6 \cdot \frac{2^n}{3^n}.$$
$$\lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{x\to+\infty} (2/3)^x = 0$$

because $0 < 2/3 < 1$. So we know

$$\lim_{n\to\infty} -6 \cdot \frac{2^n}{3^n} = 0 \lim_{n\to\infty} 6 \cdot \frac{2^n}{3^n} \qquad = 0$$

so by the Squeeze theorem, $\lim_{n\to\infty} a_n = 0$.

Problem 2 (Factorials). (a) What is 4!? What is $\frac{4!}{3!}$?

(b) What is $\frac{5!}{4!}$? What is $\frac{5!}{3!}$?

(c) Can you figure out what $\frac{202!}{200!}$ is?

Solution:

(a) $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. $\frac{4!}{3!} = \frac{24}{6} = 4$.

(b) We know $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. Then $\frac{5!}{4!} = \frac{120}{24} = 5$. But there's a better way: we have

$$\frac{5!}{4!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 5.$$

Thus we have

$$\frac{5!}{3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 5 \cdot 4 = 20.$$

(c) $\frac{202!}{200!} = 202 \cdot 201 = 40602$.

Problem 3. (a) Compute $\lim\limits_{n\to\infty} \dfrac{n}{n!}$. Justify your answer.

(b) Compute $\lim_{n\to\infty} \frac{e^n}{n!}$.

(c) Now compute $\lim_{n\to\infty} \frac{n^k}{n!}$, where $k > 0$ is an integer.

Solution:

(a)

$$\lim_{n\to\infty} \frac{n}{n!} = \lim_{n\to\infty} \frac{n}{n \cdot (n-1)!} = \lim_{n\to\infty} \frac{1}{(n-1)!} = 0.$$

If we want to justify that last limit, we can observe that $\frac{1}{(n-1)!} < \frac{1}{n}$ as long as $n > 3$, and use the squeeze theorem.

(b) For $k > 2$ we know that $e/k < 1$, so

$$\begin{aligned} \frac{e^n}{n!} &= \frac{e \cdot e \cdot e \cdot \dots \cdot e \cdot e \cdot e}{n(n-1)(n-2)\dots(3)(2)(1)} \\ &\leq \frac{e}{n} \cdot \frac{e^2}{2} \leq \frac{e^3}{n} \to 0. \end{aligned}$$

Since $0 \leq \frac{e^n}{n!} \leq \frac{e^3}{n}$ and $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{e^3}{n}$, by the squeeze theorem we know $\lim_{n\to\infty} \frac{e^n}{n!} = 0$.

(c) This one is tricky. For large k and small n this can be pretty big. But if $n > 2k$ we have

$$\begin{aligned} \frac{n^k}{n!} &= \frac{n \cdot n \cdot \dots \cdot n}{n(n-1)(n-2)\dots(3)(2)(1)} \\ &= \frac{n}{n-1} \cdot \frac{n}{n-2} \cdot \frac{n}{n-3} \cdots \frac{n}{n-k+1} \cdot \frac{1}{(n-k)!} \\ &\leq 2^k \frac{1}{(n-k)!} \leq \frac{2^k}{(n-k)}. \end{aligned}$$

But remembering k is a constant, we know that $\lim_{n\to\infty} \frac{1}{n-k} = 0$, so $\lim_{n\to\infty} \frac{2^k}{n-k} = 0$. By the squeeze theorem, $\lim_{n\to\infty} \frac{n^k}{n!} = 0$.

Problem 4. Consider the sequence $(a_n) = (\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots)$.

(a) We don't have a closed-form formula for this sequence, but we can still say things about it. What happens if we square each element of the sequence, and then divide by 2?

(b) We want to find the limit of this sequence. Half of this is easy: *if* the sequence converges, we can use a trick to find the limit.

Suppose $\lim_{n\to\infty} a_n = L$. What can you say about $L^n/2$?

(c) Can you figure out what L is, if the limit exists?

(d) That all relied on the idea that the limit existed. We want to use completeness to prove this. We need to show this sequence is increasing and bounded above.

If $0 \leq x \leq 2$, explain why $x \leq \sqrt{2x}$.

(e) If $0 \leq x \leq 2$, explain why $\sqrt{2x} \leq 2$.

(f) How does this prove the limit exists?

Solution:

(a) We get $(1, \sqrt{2}, \sqrt{2\sqrt{2}}, \dots)$, which is just our original sequence with a 1 stuck on the front.

(b) Sticking a 1 on the front of the sequence can't change the limit at all. So we see that $L^2/2 = L$.

(c) This equation is consistent if $L = 0$ or if $L = 2$. The limit clearly isn't 0, so if the limit exists it must be 2.

(d) Since $x \leq 2$, we know that $x = \sqrt{x} \cdot \sqrt{x} \leq \sqrt{2} \cdot \sqrt{x} = \sqrt{2x}$. So the sequence is increasing.

(e) Since $x \leq 2$, then $\sqrt{2x} = \sqrt{2}\sqrt{x} \leq \sqrt{2}\sqrt{2} = 2$.

(f) The sequence is increasing and bounded above. By completeness, it has a limit.

Problem 5. The discrete equivalent of a derivative is a *difference quotient*. Given a function $f(n)$ defined on positive integers, we can define $\Delta f(n) = f(n+1) - f(n)$.

(a) Does that look like a derivative? What pieces are missing, and why?

(b) If $f(n) = n^2$, compute $\Delta f(n)$. Compute $f'(n)$. How are they related?

(c) If $g(n) = \frac{1}{n}$, compute $\Delta g(n)$. Compute $g'(n)$. How are they related?

Solution:

(a) A derivative would be $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$. Here we aren't taking the limit; instead we're setting h equal to 1. This also means we don't need to explicitly write a fraction, since we're just dividing by 1.

(We could just as easily compute $\frac{f(n+3)-f(n)}{3}$, but this is simpler.)

(b)

$$\begin{aligned} \Delta f(n) &= f(n+1) - f(n) = (n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1. \\ f'(n) &= 2n. \end{aligned}$$

These are very close together! But the difference quotient is slightly different, because the function is concave up.

(c)

$$\begin{aligned}\Delta g(n) &= \frac{1}{n+1} - \frac{1}{n} = \frac{n}{n(n+1)} - \frac{n+1}{n(n+1)} = \frac{-1}{n^2+n}.\\ g'(n) &= \frac{-1}{n^2}.\end{aligned}$$

Again, these are close together. But the difference quotient is a little smaller, and a little more complex to write down.