Math 1232 Spring 2024
Single-Variable Calculus 2 Section 12
Mastery Quiz 9
Due Tuesday, April 2

This week's mastery quiz has two topics. Everyone should submit work on topic M3. If you have a 2/2 on S7, you don't need to submit it again. (Check Blackboard!)

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Secondary Topic 7: Sequences and Series

Name:

Recitation Section:

M3: Series Convergence

(a) Analyze the convergence of the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^3+n}$

Solution: We use the Ratio test. We have

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{(-2)^{n+1}/(n+1)^3+n+1}{(-2)^n/n^3+n} \right| &= \lim_{n\to\infty} \frac{2(n^3+n)}{(n+1)^3+n+1} \\ &= \lim_{n\to\infty} 2 > 1. \end{aligned}$$

This limit is greater than 1, so by the ratio test this diverges.

Alternatively, we could note that

$$\lim_{n\to\infty} \frac{(-2)^n}{n^3+n} = \pm\infty,$$

so by the divergence test this diverges. But it's a little tricky to cleanly argue that this goes to infinity; we can't really use L'Hospital's rule without getting the negative sign out of there somehow.

(b) Analyze the convergence of the series $\sum_{n=1}^{\infty} \frac{n\sin(n)}{n^3+2}$

Solution: This series has positive and negative terms, but it's not alternating. We basically have to look at absolute convergence.

We consider the series

$$\sum_{n=1}^{\infty} \left| \frac{n\sin(n)}{n^3+2} \right| = \sum_{n=1}^{\infty} \frac{n|\sin(n)|}{n^3+2}.$$

We can't really use the limit comparison test here, because the $\sin(n)$ will screw it up. But we can use the usual comparison test. We know that $0 \leq |\sin(n)| \leq 1$, so

$$\frac{n|\sin(n)|}{n^3+2} \leq \frac{n}{n^3} = \frac{1}{n^2}.$$

We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test, so this series converges by the comparison test. Thus our original series converges absolutely.

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{2n+3}$

Solution: This is an alternating series. Since the terms $\frac{\sqrt{n}}{2m+3}$ tend to zero as n goes to infinity, this converges by the alternating series test.

However, it doesn't absolutely converge. If we look at the absolute value series, we have $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n+3}$. You can see this doesn't converge in a couple ways. The integral test isn't super plausible here. You can do a comparison test to $\frac{1}{\sqrt{n}}$: this is larger than $\frac{1}{3\sqrt{n}}$ for large n, and $\frac{1}{3\sqrt{n}}$ diverges. (note: this is *not* larger than $\frac{1}{\sqrt{n}}$!)

It may be easier to use the limit comparison test, though. We have

$$\lim_{n\to\infty} \frac{\sqrt{n}/2n+3}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{n}{2n+3} = 1/2.$$

Since the series $\sum \frac{1}{\sqrt{n}}$ diverges, by the limit comparison test, $\sum \frac{\sqrt{n}}{2n+3}$ diverges, and thus our series does not converge absolutely.

S7: Sequences and Series

(a) Let $b_n = \frac{(n)!}{(n+2)!}$. Compute the first four terms of the sequence, and compute $\lim_{n\to\infty} b_n$.

Solution: $b_1 = 1/6, b_2 = 2/24 = 1/12, b_3 = 6/120 = 1/20$, and $b_4 = 24/720 = 1/30$.

We compute

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{(n+1)(n+2)} = \lim_{n\to\infty} \frac{1}{n^2+3n+2} = 0.$$

(b) Compute $\sum_{n=1}^{\infty} \frac{4}{3^{2n}}$.

Solution: This is a geometric series with $a = 4/9$ and $r = 1/9$, so we have

$$\sum_{n=1}^{\infty} \frac{4}{3^{2n}} == \frac{4/9}{1-1/9} = 1/2.$$

(c) Compute $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$.

Solution: We can do a partial fractions decomposition: we have

$$\begin{aligned} 2 &= A(n+1) + B(n+3) & \\ 2 &= 2B & \Rightarrow B = 1 \\ 2 &= -2A & \Rightarrow A = -1 \end{aligned}$$

so our sum is

$$\sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3} = \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \dots$$
$$= \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

More rigorously, we have

$$\begin{aligned}
\sum_{n=1}^{k} \frac{2}{n^2+4n+3} &= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) \\
&\quad + \dots + \left(\frac{1}{k+1} - \frac{1}{k+3}\right) \\
&= \frac{1}{2} + \frac{1}{3} - \frac{1}{k+2} - \frac{1}{k+3} \\
\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} &= \lim_{k\to\infty} \frac{1}{2} + \frac{1}{3} - \frac{1}{k+2} - \frac{1}{k+3} \\
&= \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.
\end{aligned}$$