Math 1232: Single-Variable Calculus 2
George Washington University   Spring 2024
Recitation 9

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Problem 1. Write out the first five terms of:

(a) $\sum_{k=1}^{\infty} \frac{(-2)^{k+1}}{3k}$

(b) $\sum_{k=1}^{\infty} \frac{k+1}{k!}$

(c) $\sum_{k=3}^{\infty} \frac{k+3}{k^2-k-2}$

Solution:

(a) $\frac{4}{3} - \frac{8}{6} + \frac{16}{9} - \frac{32}{12} + \frac{64}{15}$.

(b) $\frac{2}{1} + \frac{3}{2} + \frac{4}{6} + \frac{5}{24} + \frac{6}{120}$.

(c) $\frac{6}{4} + \frac{7}{10} + \frac{8}{18} + \frac{9}{28} + \frac{10}{40}$.

Problem 2. Write in series/summation notation:

(a) $1 + \frac{2}{3} + \frac{3}{5} + \frac{4}{7} + \dots$

(b) $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} + \dots$

(c) $2 + 7 + 14 + 23 + 34 + \dots$

Solution:

(a) $\sum_{k=1}^{\infty} \frac{k}{2k-1}$.

(b) $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2}$.

(c) $2 + \sum_{k=1}^{\infty} 2k + 3$.

Problem 3. (a) Use a telescoping series argument to write down a formula for $\sum_{k=1}^{n} \frac{1}{k^2+3k+2}$.

(b) Compute $\sum_{k=1}^{\infty} \frac{1}{k^2+3k+2}$.

(c) Use a telescoping series argument to write down a formula for $\sum_{k=1}^{n} \frac{2}{k^2+2k}$.

(d) Compute $\sum_{k=1}^{\infty} \frac{2}{k^2+2k}$.

(e) Use a telescoping series argument to write down a formula for $\sum_{k=1}^{n} \ln\left(\frac{k+1}{k+3}\right)$.

(f) Compute $\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k+3}\right)$.

Solution:

(a)

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k^2+3k+2} &= \sum_{k=1}^{n} \frac{1}{k+1} - \frac{1}{k+2} \\ &= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) \\ &= \frac{1}{2} - \frac{1}{n+2}.\end{aligned}$$

(b)

$$\sum_{k=1}^{\infty} \frac{1}{k^2+3k+2} = \lim_{n\to\infty} \frac{1}{2} - \frac{1}{n+2} = \frac{1}{2}.$$

(c)

$$\begin{aligned}\sum_{k=1}^{n} \frac{2}{k^2+2k} &= \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+2} \\ &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right) \\ &= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}.\end{aligned}$$

(d)
$$\sum_{k=1}^{\infty} \frac{2}{k^2+2k} = \lim_{n\to\infty} 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} = \frac{3}{2}.$$

(e)
$$\begin{aligned}\sum_{k=1}^{n} \ln\left(\frac{k+1}{k+3}\right) &= \sum_{k=1}^{n} \ln(k+1) - \ln(k+3) \\ &= \Big(\ln(2)-\ln(4)\Big) + \Big(\ln(3)-\ln(5)\Big) + \Big(\ln(4)-\ln(6)\Big) \\ &\quad + \cdots + \Big(\ln(n)-\ln(n+2)\Big) + \Big(\ln(n+1)-\ln(n+3)\Big) \\ &= \ln(2)+\ln(3)-\ln(n+2)-\ln(n+3).\end{aligned}$$

(f)
$$\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k+3}\right) = \lim_{n\to\infty} \ln(2)+\ln(3)-\ln(n+2)-\ln(n+3) = \ln(6)-\ln(n^2+5n+6) = -\infty.$$

Problem 4 (Geometric Series)**.** Compute:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{3^k}$

(b) $\displaystyle\sum_{k=2}^{\infty} \frac{(-5)^{k+2}}{2^{3k}}$

(c) $\frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \frac{5}{16} + \dots$

(d) $\frac{-2}{3} + \frac{8}{9} + \frac{-32}{27} + \dots$

(e) $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \dots$

Solution:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{3^k} = \frac{2/3}{1-2/3} = 2.$

(b) $\displaystyle\sum_{k=2}^{\infty} \frac{(-5)^{k+2}}{2^{3k}} = \frac{625/64}{1+5/8} = \frac{625/64}{13/8} = \frac{625}{104}.$

(c) $\frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \frac{5}{16} + \cdots = \sum_{k=1}^{\infty} \frac{5}{2^k} = \frac{5/2}{1-1/2} = 5.$

(d) $\frac{-2}{3} + \frac{8}{9} + \frac{-32}{27} + \cdots = \sum_{k=1}^{\infty} \frac{-2}{3} \frac{4^{k-1}}{3^{k-1}}$ and since the ratio $r = \frac{4}{3} > 1$ this series diverges.

(e) $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{3^k} = \frac{1/3}{1+1/3} = \frac{1/3}{4/3} = \frac{1}{4}$.

Problem 5 (Infinite Decimals)**.** We want to find a rational representation of the infinite decimal $0.\overline{47}$. That is, we want to write $0.\overline{47} = \frac{p}{q}$ for integers p, q.

(a) First, what happens if we multiply $0.\overline{47}$ by 100?

(b) Using part (a), what can you tell about $(99) \cdot 0.\overline{47}$?

(c) Give a rational representation of $0.\overline{47}$.

(d) Now let's take a different approach. Write $0.\overline{47}$ as an infinite series.

(e) What kind of series is this? Can you use that fact to find a rational representation of $0.\overline{47}$?

(f) Now use the same logic to find a rational representation of $2.\overline{63}$.

Solution:

(a) $0.\overline{47} \cdot 100 = 47.\overline{47}$.

(b)

$$\begin{aligned} 99 \cdot 0.\overline{47} &= 100 \cdot 0.\overline{47} - 0.\overline{47} \\ &= 47.\overline{47} - 0.\overline{47} = 47 \end{aligned}$$

(c) Thus $0.\overline{47} = \frac{47}{99}$.

(d) We can write $0.\overline{47} = \sum_{k=1}^{\infty} 47 \cdot 100^{-k}$.

(e) This is a geometric series with $a = \frac{47}{100}$ and $r = \frac{1}{100}$. Thus

$$0.\overline{47} = \sum_{k=1}^{\infty} 47 \cdot 100^{-k} = \frac{47/100}{1 - 1/100} = \frac{47/100}{99/100} = \frac{47}{99}.$$

(f) We can ignore the 2 until later. We can see

$$\begin{aligned} 0.\overline{63} &= \sum_{k=1}^{\infty} 63 \cdot 100^{-k} \\ &= \frac{63/100}{1 - 1/100} = \frac{63/100}{99/100} \\ &= \frac{63}{99} \\ 2.\overline{63} &= 2 + \frac{63}{99} = \frac{261}{99}. \end{aligned}$$