Jefferson's Method

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October 8, 2025

Definition (Jefferson's method)

- Choose a modified divisor d
- Compute the modified quotas p_k/d
- Round these down to obtain $a_k = \lfloor p_k/d \rfloor$.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have the Jefferson apportionment.
- Otherwise, choose a new d and try again.

Jefferson's Method

Proposition

Suppose h, n, and $p_1, \ldots, p_n$ are given as inputs to our apportionment function. If d and d' are two different divisors, yielding Jefferson apportionments $a_1, \ldots, a_n$ and $a'_1, \ldots, a'_n$ respectively, then $a_k = a'_k$ for each state k.

Proof.

- Suppose "without loss of generality" that $d \leq d'$.
- For every state, $p_k/d \geq p_k/d'$.
- Rounding down doesn't change that, so $a_k \geq a'_k$.
- Adding up won't change that:

  $a_1 + a_2 + \cdots + a_n \geq a'_1 + a'_2 + \cdots + a'_n$.

Jefferson's Method

Proposition

Suppose $h, n,$ and $p_1, \ldots, p_n$ are given as inputs to our apportionment function. If d and d' are two different divisors, yielding Jefferson apportionments $a_1, \ldots, a_n$ and $a'_1, \ldots, a'_n$ respectively, then $a_k = a'_k$ for each state k.

Proof.

- For every state, $a_k \geq a'_k$
- $a_1 + a_2 + \cdots + a_n \geq a'_1 + a'_2 + \cdots + a'_n$
- $h = a_1 + a_2 + \cdots + a_n \geq a'_1 + a'_2 + \cdots + a'_n = h$
- Totals must be the same
- Therefore, for each k, $a_k = a'_k$.

$\square$

Critical Divisors

How do we find the apportionment?

- Trial and error
- Binary search
- But there's a better way!

When does the apportionment change?

- Small changes in d don't usually matter
- Find specific values where something changes
- When does a state get a new representative?

Critical Divisors

- Had $p_3 = 5{,}300{,}000$
- If $d = 850{,}000$ then $q = \frac{5{,}300{,}000}{850{,}000} \approx 6.24$.
- 6.24 "rounds down to" 6. What does that mean?
- $6 \leq 6.24 < 7$
- $6 \leq \frac{5{,}300{,}000}{850{,}000} < 7$
- $850,000 \leq \frac{5{,}300{,}000}{6}$ but $\frac{5{,}300{,}000}{7} < 850,000$.
- When does something interesting happen? When $d = \frac{5{,}300{,}000}{6}$ or $d = \frac{5{,}300{,}000}{7}$.

Definition

We call a number of the form $\frac{p_k}{m}$ for a positive integer k a *(Jefferson) critical divisor* for the state k.

Critical Divisors

Definition

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Computing with Critical Divisors

- Imagine d is very very large.
- Every state gets no representatives
- Increase d slowly
- A state will get a new representative when d crosses a critical divisor
- So we give seats to the biggest h critical divisors
- Can pick any d between divisors h and $h+1$.

Example

Find the Jefferson apportionment with $n = 3, h = 10$, and state populations $p_1 = 1{,}500{,}000$, $p_2 = 3{,}200{,}000$, $p_3 = 5{,}300{,}000$.

d	State 1	State 2	State 3
1	$\frac{1{,}500{,}000}{1} = 1,500,000$	$\frac{3{,}200{,}000}{1} = 3,200,000$	$\frac{5{,}300{,}000}{1} = 5,300,000$
2	$\frac{1{,}500{,}000}{2} = 750,000$	$\frac{3{,}200{,}000}{2} = 1,600,000$	$\frac{5{,}300{,}000}{2} = 2,650,000$
3	$\frac{1{,}500{,}000}{3} = 500,000$	$\frac{3{,}200{,}000}{3} = 1,066,667$	$\frac{5{,}300{,}000}{3} = 1,766,667$
4	$\frac{1{,}500{,}000}{4} = 375,000$	$\frac{3{,}200{,}000}{4} = 800,000$	$\frac{5{,}300{,}000}{4} = 1,325,000$
5	$\frac{1{,}500{,}000}{5} = 300,000$	$\frac{3{,}200{,}000}{5} = 640,000$	$\frac{5{,}300{,}000}{5} = 1,060,000$
6	$\frac{1{,}500{,}000}{6} = 250,000$	$\frac{3{,}200{,}000}{6} = 533,333$	$\frac{5{,}300{,}000}{6} = 883,333$
7	$\frac{1{,}500{,}000}{7} = 214,857$	$\frac{3{,}200{,}000}{7} = 457,143$	$\frac{5{,}300{,}000}{7} = 757,143$

Critical Divisors

Advantages and Disadvantages

- No trial and error
- Straightforward calculation
- A *lot* of straightforward calculation

Can we speed this up?

- Can calculate s easily
- d always bigger
- Count up from there

Critical Divisors

Example

If $n = 3$, $h = 10$, $p_1 = 1{,}500{,}000$, $p_2 = 3{,}200{,}000$, and $p_3 = 5{,}300{,}000$ then $p = 10{,}000{,}000$ and $s = 1{,}000{,}000$

d	State 1	State 2	State 3
1	$\frac{1{,}500{,}000}{1} = 1,500,000$	$\frac{3{,}200{,}000}{1} = 3,200,000$	$\frac{5{,}300{,}000}{1} = 5,300,000$
2	$\frac{1{,}500{,}000}{2} = 750,000$	$\frac{3{,}200{,}000}{2} = 1,600,000$	$\frac{5{,}300{,}000}{2} = 2,650,000$
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Critical Divisors

- Even better: don't have to calculate them all
- Start from standard divisor s
- Find round-down apportionments $\lfloor q_k \rfloor$
- The *next* critical divisor will be $\frac{p_k}{\lfloor q_k \rfloor + 1}$.

Critical Divisors

Example

If $n = 3, h = 10$, $p_1 = 1{,}500{,}000$, $p_2 = 3{,}200{,}000$, and $p_3 = 5{,}300{,}000$ then $p = 10{,}000{,}000$ and $s = 1{,}000{,}000$

		$s = 1,000,000$		Next Divisor	$d = 850,000$	
k	p_k	q_k	$\lfloor q_k \rfloor$	$\frac{p_k}{\lfloor q_k \rfloor + 1}$	q	a_k
1	1,500,000	1.50	1	750,000	1.76	1
2	3,200,000	3.20	3	800,000	3.76	3
3	5,300,000	5.30	5	883,333	6.24	6
Total	10,000,000		9			10

Properties of Jefferson's Method

- Favors large states relative to Hamilton's method

Example

Find the Hamilton and Jefferson apportionments when $n = 2, h = 10$, with $p_1 = 1{,}800{,}000$ and $p_2 = 8{,}200{,}000$.

k	p_k	q_k	$\lfloor q_k \rfloor$	Ham a_k	CD	$d = 910{,}000$	Jef a_k
1	1,800,000	1.8	1	2	900,000	1.98	1
2	8,200,000	8.2	8	8	911,111	9.02	9

Properties of Jefferson's Method

Example

Find the Hamilton and Jefferson apportionments when

$n = 4$, $h = 10$, with populations below:

k	p_k	q_k	$\lfloor q_k \rfloor$	Ham a_k	CD	$d = 725{,}000$	Jeff a_k
1	1,500,000	1.5	1	2	750,000	2.07	2
2	1,400,000	1.4	1	1	700,000	1.93	1
3	1,300,000	1.3	1	1	650,000	1.79	1
4	5,800,000	5.8	5	6	966,667	8	8
	10,000,000		8	10			12 ??

- That didn't work
- What went wrong?

Properties of Jefferson's Method

Example

Find the Hamilton and Jefferson apportionments when

$n = 4$, $h = 10$, with populations below:

k	p_k	q_k	$\lfloor q_k \rfloor$	Ham a_k	CD		CD	
1	1,500,000	1.5	1	2	750,000	1	750,000	1
2	1,400,000	1.4	1	1	700,000	1	700,000	1
3	1,300,000	1.3	1	1	650,000	1	700,000	1
4	5,800,000	5.8	5	6	966,667	7	828,571	8
	10,000,000		8	10		9		10

Example

Find the Hamilton and Jefferson apportionments when

$n = 4$, $h = 10$, with populations below:

k	p_k	q_k	$\lfloor q_k \rfloor$	Ham a_k	CD	$d = 800{,}000$	Jeff a_k
1	1,500,000	1.5	1	2	750,000	1.88	1
2	1,400,000	1.4	1	1	700,000	1.75	1
3	1,300,000	1.3	1	1	650,000	1.62	1
4	5,800,000	5.8	5	6	966,667	7.25	7
	10,000,000		8	10			10

Properties of Jefferson's Method

Important Computational Note

- Hamilton's method rounds states up, one by one
- Jefferson's method doesn't do that
- Some states can get bumped up more than once
- To use critical divisors:
  - Compute critical divisors
  - Give seat to state with largest critical divisor
  - *Compute critical divisors again*
  - Allocate seat again
  - Repeat until all excess seats are allocated.
- But also we get a weird result here, right?

Quota Rules

Definition

We say it's a quota violation if an apportionment method gives a state more representatives than its upper quota, or less than its lower quota.

An apportionment method satisfies the quota rule if it assigns every state either its lower quota or its upper quota.

Proposition

Jefferson's method violates the quota rule.

- Sometimes useful to split this into two ideas.

Quota Rules

Definition

An apportionment method satisfies the *upper quota rule* if it never assigns a state more than its upper quota.

A violation of this rules is an *upper quota violation*.

Definition

An apportionment method satisfies the *lower quota rule* if it never assigns a state less than its lower quota.

A violation of this rules is an *lower quota violation*.

Proposition

Jefferson's method violates the upper quota rule.

Quota Rules

Proposition

Jefferson's method satisfies the lower quota rule.

Proof.

- Jefferson apportionment with $d = s$ gives every state $\lfloor q_k \rfloor$
- That won't allocate enough seats
- Need to pick a smaller d
- That will never give any state *fewer* seats than s would
- Every state gets at least its lower quota.

$\square$

Jefferson and Monotonicity

Proposition

Jefferson's method is house monotone.

Proof.

Next time! $\square$