

Divisor Methods

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Jefferson's Method

Definition (Jefferson's method)

- Choose a modified divisor d
- Compute the modified quotas p_k/d
- Round these down to obtain $a_k = \lfloor p_k/d \rfloor$.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have the Jefferson apportionment.
- Otherwise, choose a new d and try again.

Jefferson and Monotonicity

Proposition

Jefferson's method is house monotone.

Proof 1 with Modified Divisors.

- Increasing h will require a smaller divisor d
- Decreasing d gives a higher p_k/d to each state
- Rounding down a larger number will never give a smaller number.
- (It might not give a bigger number either but that's fine.)
- So reducing d will never cause a state to lose a seat
- Therefore increasing h will never cause a state to lose a seat.



Jefferson and Monotonicity

Proposition

Jefferson's method is house monotone.

Proof 2 with Critical Divisors.

- List all critical divisors in decreasing order
- We allocate h seats by choosing the first h divisors in the list
- If we increase h , we get more divisors
- But this will include all the divisors we got the first time
- We allocate all the original seats, and then allocate more
- Therefore no state will lose a seat when we increase h .



Adams's Method

- Do we have to round down?

Definition (Adams's method)

- Choose a modified divisor d
- Compute the modified quotas p_k/d
- Round these *up* to obtain $a_k = \lceil p_k/d \rceil$.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have the Adams apportionment.
- Otherwise, choose a new d and try again.

Adams's Method

Example

Find the Jefferson and Adams apportionments when $n = 2$, $h = 10$, with $p_1 = 1,800,000$ and $p_2 = 8,200,000$.

- We've seen Jefferson already
- Can use guess-and-check, binary search, or critical divisors

k	p_k	q_k	$\lfloor q_k \rfloor$	Ham	CD	$d = 910,000$	Jef a_k
1	1,800,000	1.8	1	2	900,000	1.98	1
2	8,200,000	8.2	8	8	911,111	9.02	9

Adams's Method

Example

Find the Jefferson and Adams apportionments when $n = 2$, $h = 10$, with $p_1 = 1,800,000$ and $p_2 = 8,200,000$.

		$s = 1,000,000$			$d = 910,000$			$d = 1,100,000$		
k	p_k	q_k	$\lfloor q_k \rfloor$	$\lceil q_k \rceil$	q	$\lfloor q \rfloor$	$\lceil q \rceil$	q	$\lfloor q \rfloor$	$\lceil q \rceil$
1	1,800,000	1.8	1	2	1.98	1	2	1.63	1	2
2	8,200,000	8.2	8	9	9.02	9	10	7.45	7	8
			9	11		10	12		8	10

Adams's Method

- Can also use a critical divisors approach
- But what happens when d is very big?

d	State 1	State 2
1	$\frac{1,800,000}{1} = 1,800,000$	$\frac{8,200,000}{1} = 8,200,000$
2	$\frac{1,800,000}{2} = 900,000$	$\frac{8,200,000}{2} = 4,100,000$
3	$\frac{1,800,000}{3} = 600,000$	$\frac{8,200,000}{3} = 2,733,333$
4	$\frac{1,800,000}{4} = 450,000$	$\frac{8,200,000}{4} = 2,050,000$
5	$\frac{1,800,000}{5} = 360,000$	$\frac{8,200,000}{5} = 1,640,000$
6	$\frac{1,800,000}{6} = 300,000$	$\frac{8,200,000}{6} = 1,366,667$
7	$\frac{1,800,000}{7} = 257,142$	$\frac{8,200,000}{7} = 1,171,429$
8	$\frac{1,800,000}{8} = 225,000$	$\frac{8,200,000}{8} = 1,025,000$
9	$\frac{1,800,000}{9} = 200,000$	$\frac{8,200,000}{9} = 911,111$

Adams's Method

- Can also use a critical divisors approach
- But what happens when d is very big?

d	State 1	State 2
0	∞	∞
1	$\frac{1,800,000}{1} = 1,800,000$	$\frac{8,200,000}{1} = 8,200,000$
2	$\frac{1,800,000}{2} = 900,000$	$\frac{8,200,000}{2} = 4,100,000$
3	$\frac{1,800,000}{3} = 600,000$	$\frac{8,200,000}{3} = 2,733,333$
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7	$\frac{1,800,000}{7} = 257,142$	$\frac{8,200,000}{7} = 1,171,429$
8	$\frac{1,800,000}{8} = 225,000$	$\frac{8,200,000}{8} = 1,025,000$

Adams's Method

- $d = s$ will be too *low*
- Need to slowly *increase* it to find the right divisor

k	p_k	q_k	$\lfloor q_k \rfloor$	$\frac{p_k}{\lfloor q_k \rfloor + 1}$	$\frac{p_k}{\lfloor q_k \rfloor}$	$d = 1,100,000$	Adams
1	1,800,000	1.8	1	900,000	1,800,000	1.63	2
2	8,200,000	8.2	8	911,111	1,025,000	7.45	8

Adams's Method

Example

Find the Hamilton, Jefferson and Adams apportionments when $n = 2$, $h = 10$, with $p_1 = 1,200,000$ and $p_2 = 8,800,000$.

k	p_k	q_k	$\lfloor q_k \rfloor$	Ham	$\frac{p_k}{\lfloor q_k \rfloor + 1}$	$d = 900,000$	Jef
1	1,200,000	1.2	1	1	600,000	1.33	1
2	8,800,000	8.8	8	9	977,778	9.78	9

k	p_k	q_k	$\lfloor q_k \rfloor$	$\lceil q_k \rceil$	$\frac{p_k}{\lceil q_k \rceil}$	$d = 1,150,000$	Adams
1	1,200,000	1.2	1	2	1,200,000	1.04	2
2	8,800,000	8.8	8	9	1,100,000	7.65	8

Adams's Method

- Adams's method favors small states
- Adams's method automatically guarantees each state at least one seat.

Proposition

Adams's method violates the lower quota rule.

Proof.

- Proof by example
- Not hard to find a case where a large state gets less than its lower quota.



Splitting the difference: Webster's Method

- Jefferson's method rounds down, favors large states
- Adams's method rounds up, favors small states
- Split the difference and round normally?

Definition (Webster's method)

- Choose a modified divisor d
- Compute the modified quotas p_k/d
- Round these to the nearest whole number to obtain a_k .
- We've been calling this “grade-school rounding”; a fancier name is **arithmetic rounding**.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have the Webster apportionment.
- Otherwise, choose a new d and try again.

Webster's Method

- In Jefferson's method, s is always too big.
- In Adams's method, s is always too small.
- In Webster's method, it could be too big, or too small, or just right.
- Think about critical divisors bigger *and* smaller.

Which divisors are critical?

- Whole numbers aren't the important ones
- Round down at 3.49 but up at 3.5
- Want to look at $\frac{p}{m+1/2}$.

Discussion Question

- What number is halfway between 2 and 4?
- What number is halfway between 1 and 100?
- What number is halfway between 1 and $\frac{1}{100}$?
- What does “halfway” mean?