Divisor Methods

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Jefferson's Method

Definition (Jefferson's method)

- Choose a modified divisor d
- Compute the modified quotas p_k/d
- Round these down to obtain $a_k = \lfloor p_k/d \rfloor$.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have the Jefferson apportionment.
- Otherwise, choose a new d and try again.

Adams's Method

Definition (Adams's method)

- Choose a modified divisor d
- Compute the modified quotas p_k/d
- Round these *up* to obtain $a_k = \lceil p_k/d \rceil$.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have the Adams apportionment.
- Otherwise, choose a new d and try again.

Splitting the difference: Webster's Method

- Jefferson's method rounds down, favors large states
- Adams's method rounds up, favors small states
- Split the difference and round normally?

Definition (Webster's method)

- Choose a modified divisor d
- Compute the modified quotas p_k/d
- Round these to the nearest whole number to obtain a_k.
- We've been calling this "grade-school rounding"; a fancier name is arithmetic rounding.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have the Webster apportionment.
- Otherwise, choose a new d and try again.

Webster's Method

- In Jefferson's method, s is always too big.
- In Adams's method, s is always too small.
- In Webster's method, it could be too big, or too small, or just right.
- Think about critical divisors bigger *and* smaller.

Which divisors are critical?

- Whole numbers aren't the important ones
- Round down at 3.49 but up at 3.5
- Want to look at $\frac{p}{m+1/2}$.

Example

Apportion $h = 10$ seats to $n = 3$ states with populations 3,300, 5,100, and 1,600, using the methods of Hamilton and Jefferson.

		$s = 1{,}000$			Jefferson $d = 840$		
k	p_k	q_K	$\lfloor q_k \rfloor$	Ham	CD $\frac{p_k}{\lfloor q_k \rfloor + 1}$	Jef q	Jef a_k
1	3,300	3.3	3	3	825	3.93	3
2	5,100	5.1	5	5	850	6.07	6
3	1,600	1.6	1	2	800	1.90	1

Webster's Method

Example

Apportion $h = 10$ seats to $n = 3$ states with populations 3,300, 5,100, and 1,600, using the method of Adams.

		$s = 1{,}000$			Adams $d = 1{,}150$		
k	p_k	q_K	$\lfloor q_k \rfloor$	Ham	CD $\frac{p_k}{\lfloor q_k \rfloor}$	Adams q	Adams a_k
1	3,300	3.3	3	3	1100	2.86	3
2	5,100	5.1	5	5	1020	4.43	5
3	1,600	1.6	1	2	1600	1.39	2

Webster's Method

Example

Apportion $h = 10$ seats to $n = 3$ states with populations 3,300, 5,100, and 1,600, using the method of Webster.

		$s = 1{,}000$			Webster $d = 1{,}000$			
k	p_k	q_K	$\lfloor q_k \rfloor$	Ham	$\frac{p_k}{\lfloor q_k \rfloor + 1/2}$	$\frac{p_k}{\lfloor q_k \rfloor - 1/2}$	q	a_k
1	3,300	3.3	3	3	943	1,320	3.3	3
2	5,100	5.1	5	5	927	1,133	5.1	5
3	1,600	1.6	1	2	1,067	3,200	1.6	2

Rounding

Definition

A rounding function is a function that takes in a real number, outputs an integer, and has the following two properties:

1. If x is an integer, then $f(x) = x$.
2. If $x > y$, then $f(x) \geq f(y)$.

That is:

- Every integer rounds to itself
- Rounding will never take the bigger number and make it smaller.

Divisor Methods

Definition

A divisor method is an apportionment method that works as follows.

- Choose a rounding function f.
- Choose a modified divisor d
- Compute the modified quotas p_k/d
- Round these according to our chosen rounding function to obtain $a_k = f(p_k/d)$.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have our apportionment.
- Otherwise, choose a new d and try again.

Rounding

Rounding Functions

- Rounding down: $\lfloor x \rfloor$ Jefferson
- Rounding up: $\lceil x \rceil$ Adams
- "Grade school" or arithmetic rounding Webster

Arithmetic rounding

- Find the number halfway between two integers
- Round up if you're above it, and down if you're below it
- But what do we mean by "half"?

Rounding

Discussion Question

- What number is halfway between 2 and 4?
- What number is halfway between 1 and 100?
- What number is halfway between $\frac{1}{2}$ and $\frac{1}{4}$?
- What does "halfway" mean?

Averages

Definition

The arithmetic mean of two numbers m and n is $\frac{m+n}{2}$.

Definition

The geometric mean of two numbers m and n is $\sqrt{mn}$.

Example

- The arithmetic mean of 2 and 4 is $\frac{2+4}{2} = \frac{6}{2} = 3$
- The geometric mean of 2 and 4 is $\sqrt{2 \cdot 4} = \sqrt{8} = 2.83$
- Both of these are halfway between 2 and 4, for some definition of "halfway".

Geometric Averages

Example

- The arithmetic mean of 1 and 100 is $\frac{100+1}{2} = \frac{101}{2} = 50.5$
- The geometric mean of 1 and 100 is $\sqrt{1 \cdot 100} = \sqrt{100} = 10$
- Both of these are halfway between 1 and 100, for some definition of "halfway".

Geometric mean

- Good when we want to think about ratios
- Good when numbers are very different sizes from each other

Geometric Rounding

0 to 1	1 to 2	2 to 3	3 to 4	4 to 5	5 to 6	6 to 7	7 to 8
$\sqrt{0}$	$\sqrt{2}$	$\sqrt{6}$	$\sqrt{12}$	$\sqrt{20}$	$\sqrt{30}$	$\sqrt{42}$	$\sqrt{56}$
0	1.414	2.449	3.464	4.472	5.477	6.481	7.483

Definition

- If a number is between m and $m+1$, we can round geometrically by rounding it up if it's above the geometric mean $\sqrt{m(m+1)}$, and down if it's below the geometric mean.
- Useful cutoffs are given in the table above.

Hill's Method

Definition

Hill's method is the apportionment method that works as follows.

- Choose a modified divisor d
- Compute the modified quotas p_k/d
- Round these modified quotas geometrically to the geometrically-nearest integer to obtain a_k.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have our apportionment.
- Otherwise, choose a new d and try again.

Hill's Method

Joseph Hill

- Chief statistician at the Census Bureau (1909–1921)
- Hill's method adopted by Congress in 1941
- We still use it today

The Harmonic Mean

Discussion Question

- What number is halfway between $\frac{1}{2}$ and $\frac{1}{4}$?

- Arithmetic: $\frac{1/2+1/4}{2} = \frac{3/4}{2} = \frac{3}{8} = 0.375$
- Geometric: $\sqrt{(1/2)(1/4)} = \sqrt{1/8} = \frac{\sqrt{2}}{4} \approx 0.3535$
- Intuitively nice-feeling: $1/3 \approx 0.333$.

The Harmonic Mean

Definition

- The harmonic mean of two numbers is the reciprocal of the average of their reciprocals.
- The harmonic mean of x and y is $\dfrac{1}{\left(\frac{1/x+1/y}{2}\right)}$
- Formula looks awful but it's doing something useful.

Example

The harmonic mean of $1/2$ and $1/4$ is $\frac{1}{\left(\frac{2+4}{2}\right)} = \frac{1}{3}$.

A nicer-looking formula

$$\frac{1}{\left(\frac{1/x+1/y}{2}\right)} = \frac{2}{\frac{1}{x}+\frac{1}{y}} = \frac{2}{\frac{y+x}{xy}} = \frac{2xy}{x+y}.$$

Harmonic Rounding

0 to 1	1 to 2	2 to 3	3 to 4	4 to 5	5 to 6	6 to 7	7 to 8
$\frac{0}{1}$	$\frac{4}{3}$	$\frac{12}{5}$	$\frac{24}{7}$	$\frac{40}{9}$	$\frac{60}{11}$	$\frac{84}{13}$	$\frac{112}{15}$
0	1.333	2.400	3.429	4.444	5.455	6.462	7.467

Definition

- If a number is between m and $m+1$, we can round harmonically by rounding it up if it's above the harmonic mean $\frac{2m(m+1)}{2m+1}$, and down if it's below the harmonic mean.
- Useful cutoffs are given in the table above.

Dean's Method

Definition

Dean's method is the apportionment method that works as follows.

- Choose a modified divisor d
- Compute the modified quotas p_k/d
- Round these modified quotas harmonically to the harmonically-nearest integer to obtain a_k.
- If $a_1 + a_2 + \cdots + a_n = h$, then we have our apportionment.
- Otherwise, choose a new d and try again.

- James Dean was a math professor at the University of Vermont in the early 1800s
- Dean's method was considered in 1830, but never used.

Comparison of rounding methods

	Rounding Function and Method				
	Round Up	Harmonic	Geometric	Arithmetic	Round Down
	Adams	Dean	Hill	Webster	Jefferson
0–1	0	0	0	0.5	1
1–2	1	1.333	1.414	1.5	2
2–3	2	2.400	2.449	2.5	3
3–4	3	3.429	3.464	3.5	4
4–5	4	4.444	4.472	4.5	5
5–6	5	5.455	5.477	5.5	6
6–7	6	6.462	6.481	6.5	7
7–8	7	7.467	7.484	7.5	8

Webster, Dean, and Hill

Example

Apportion $h = 10$ seats to $n = 3$ states with populations 1,385, 2,390, and 6,225, using Hamilton, Webster, Hill, and Dean.

		$s = 1{,}000$			$d = 1{,}000$			
k	p_k	q_K	$\lfloor q_k \rfloor$	Ham	Quota	Webster	Hill	Dean
1	1,385	1.385	1	1	1.385	1	1	2
2	2,390	2.390	2	3	2.390	2	2	2
3	6,225	6.225	6	6	6.225	6	6	6
	10,000		9	10		9	9	10 ✓

Example

Apportion $h = 10$ seats to $n = 3$ states with populations 1,385, 2,390, and 6,225, using Hamilton, Webster, Hill, and Dean.

		$s = 1{,}000$			$d = 950$			
k	p_k	q_K	$\lfloor q_k \rfloor$	Ham	Quota	Webster	Hill	Dean
1	1,385	1.385	1	1	1.458	1	2	2
2	2,390	2.390	2	3	2.516	3	3	3
3	6,225	6.225	6	6	6.553	7	7	7
	10,000		9	10		11	12	12 ✓

Critical Divisors

When do critical divisors happen?

- Adams and Jefferson: Tipping points at whole numbers.
- Critical divisors at p_k/m.
- Webster: tipping points at exact halves.
- Critical divisors at $\frac{p_k}{m+1/2}$.
- Hill: Tipping points at $\sqrt{m(m+1)}$.
- Critical divisors at $\dfrac{p_k}{\sqrt{m(m+1)}}$.
- Dean: Tipping points at $\frac{2m(m+1)}{2m+1}$.
- Critical divisors at $\dfrac{p_k}{\left(\frac{2m(m+1)}{2m+1}\right)} = \dfrac{p_k(2m+1)}{2m(m+1)}$.

Critical Divisors

Method	Critical divisor for state k
Adams	$\dfrac{p_k}{a_k}$
Dean	$\dfrac{p_k(2(a_k+1))}{2a_k(a_k+1)}$
Hill	$\dfrac{p_k}{\sqrt{a_k(a_k+1)}}$
Webster	$\dfrac{p_k}{(a_k+1/2)}$
Jefferson	$\dfrac{p_k}{a_k+1}$

Webster critical divisors

Example

Apportion $h = 10$ seats to $n = 3$ states with populations 1,385, 2,390, and 6,225, using Webster's method.

		$s = 1{,}000$				$d = 957$	
k	p_k	q_K	$\lfloor q_k \rfloor$	$\frac{p_k}{\lfloor q_k \rfloor - 1/2}$	$\frac{p_k}{\lfloor q_k \rfloor + 1/2}$	Quota	Webster
1	1,385	1.385	1	2770	923	1.447	1
2	2,390	2.390	2	1593	956	2.497	2
3	6,225	6.225	6	1132	958	6.501	7
	10,000		9				10 ✓

Hill critical divisors

Example

Apportion $h = 10$ seats to $n = 3$ states with populations 1,385, 2,390, and 6,225, using Hill's method.

		$s = 1{,}000$				$d = 978$	
k	p_k	q_K	$\lfloor q_k \rfloor$	$\frac{p_k}{\sqrt{\lfloor q_k \rfloor (\lfloor q_k \rfloor \pm 1)}}$		Quota	Hill
1	1,385	1.385	1	∞	979	1.416	2
2	2,390	2.390	2	1690	976	2.444	2
3	6,225	6.225	6	1137	961	6.365	6
	10,000		9				10 ✓

Dean critical divisors

Example

Apportion $h = 10$ seats to $n = 3$ states with populations 1,385, 2,390, and 6,225, using Dean's method.

		$s = 1{,}000$				$d = 1{,}000$	
k	p_k	q_K	$\lfloor q_k \rfloor$	$\frac{p_k(2\lfloor q_k \rfloor + 1)}{2\lfloor q_k \rfloor(\lfloor q_k \rfloor \pm 1)}$		Quota	Dean
1	1,385	1.385	1	∞	1038	1.385	2
2	2,390	2.390	2	2988	996	2.390	2
3	6,225	6.225	6	1348	963	6.225	6
	10,000		9				10 ✓

A Table of Hill Critical Divisors

m	State 1	State 2	State 3
0	$\frac{1{,}385}{0} = \infty$	$\frac{2{,}390}{0} = \infty$	$\frac{6{,}225}{0} = \infty$
1	$\frac{1{,}385}{\sqrt{2}} = 979$	$\frac{2{,}390}{\sqrt{2}} = 1,690$	$\frac{6{,}225}{\sqrt{2}} = 4,402$
2	$\frac{1{,}385}{\sqrt{6}} = 565$	$\frac{2{,}390}{\sqrt{6}} = 976$	$\frac{6{,}225}{\sqrt{6}} = 2,541$
3	$\frac{1{,}385}{\sqrt{12}} = 400$	$\frac{2{,}390}{\sqrt{12}} = 690$	$\frac{6{,}225}{\sqrt{12}} = 1,797$
4	$\frac{1{,}385}{\sqrt{20}} = 310$	$\frac{2{,}390}{\sqrt{20}} = 690$	$\frac{6{,}225}{\sqrt{20}} = 1,392$
5	$\frac{1{,}385}{\sqrt{30}} = 253$	$\frac{2{,}390}{\sqrt{30}} = 436$	$\frac{6{,}225}{\sqrt{30}} = 1,137$
6	$\frac{1{,}385}{\sqrt{42}} = 214$	$\frac{2{,}390}{\sqrt{42}} = 369$	$\frac{6{,}225}{\sqrt{42}} = 961$