Criteria for Evaluating Divisor Methods

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Neutrality

Definition

An apportionment method is neutral if permuting the populations of states permutes the resulting numbers of seats in the same way.

- Can still have bias towards large or small states
- Can't have a (formal) bias towards Western states, or conservative states, or states whose names start with a vowel

Discussion Question

What aspect of our current apportionment system isn't neutral?

Proportionality

Definition

An apportionment method is proportional if it produces the same result for two censuses with the same house size, and the same relative populations p_k/p.

- Idea:If every state doubles, that shouldn't change the apportionment

Definition

We say the population distribution of a census is the list $p_1/p, p_2/p, \ldots, p_n/p$.

- Tells you what fraction of the total population each state has
- Proportional methods depend only on the population distribution

Proportionality

Proposition

Hamilton's method is proportional.

Proof.

- Hamilton's method depends only on the standard quotas $q_k = p_k/s$.
- The standard divisor $s = p/h$
- Can rewrite this as $q_k = \frac{p_k}{p/h} = h\frac{p_k}{p}$
- Standard quota depends only on the house size h and the population distribution p_k/p.

□

Proportionality

Proposition

All divisor methods are proportional.

Proof by Modified Divisors.

- Method works by dividing p_k by a divisor d
- If every population increases by a factor of c can increase divisor to cd
- Modified quotas will be $\frac{cp_k}{cd} = \frac{p_k}{d}$.
- Get the same modified quotas with the same rounded results.

□

Proportionality

Proposition

All divisor methods are proportional.

Proof by Critical Divisors.

- Method works by writing down the critical divisors from largest to smallest, and taking the h largest
- Critical divisors are $\frac{p_k}{f(m)}$ for some rounding function m
- Instead can compute $\frac{p_k/p}{f(m)}$
- This just divides our entire list by p without changing the order
- The list, and thus the result, only depends on the population distribution.

□

Order-Preserving

Definition

An apportionment method is order-preserving if, whenever $a_i > a_j$, then $p_i > p_j$.

- Bigger states should never get fewer seats
- We really really want this one
- All our methods so far do have it.

Remark

It would be reasonable and logical to call this the "population monotone" property. But we've already used that term for something else.

Quota Rules

Definition

- We say it's a quota violation if an apportionment method gives a state more representatives than its upper quota, or less than its lower quota.
- An apportionment method satisfies the quota rule if it assigns every state either its lower quota or its upper quota.
- An apportionment method satisfies the upper quota rule if it never assigns a state more than its upper quota.
- A violation of this rule is an upper quota violation.
- An apportionment method satisfies the lower quota rule if it never assigns a state less than its lower quota.
- A violation of this rule is an lower quota violation.

House Monotonicity

Definition

An apportionment method is called house monotone if an increase in h, while all other parameters remain the same, can never cause any seat allocation a_k to decrease.

- Saw in an early example that Hamilton's method is not house monotone
- Alabama paradox in 1880 shows Hamilton's method is not house monotone *in practice*.

Exercise

Lowndes's method is not house monotone.

House Monotonicity

Proposition

All divisor methods are house monotone.

Proof.

- Consider any divisor method with rounding function f
- Suppose divisor d apportions exactly h seats
- If we want to increase h, we will need to decrease d
- Get a larger modified quota p_k/d' for each state
- By definition, rounding a larger number won't give a smaller number
- That is, since $p_k/d' > p_k/d$, we know $f(p_k/d') \geq f(p_k/d)$
- Therefore no state will get a smaller apportionment.

□

House Monotonicity

Proposition

All divisor methods are house monotone.

Corollary

Hamilton's method is not a divisor method.

- More interesting than it sounds
- We didn't describe Hamilton's method as a divisor method
- This results says we *cannot* describe Hamilton's method with *any* rounding function.

Population Monotonicity

Definition

- A method is called population monotone if a state can never lose a seat when its population increases while no other state's population increases.
- In algebraic terms, whenever $a_i' < a_i$ and $a_j' > a_j$, it must be the case either that $p_i' < p_i$ or $p_j' > p_j$.

Proposition

Hamilton's method isn't population monotone.

Exercise

Lowndes's method isn't population monotone.

Population Monotonicity

Proposition

All divisor methods are population monotone.

Proof.

- Suppose $a_i' < a_i$ and $a_j' > a_j$.
- Want to show: either $p_i' < p_i$ or $p_j' > p_j$.
- (Stop and think about what those algebraic sentences mean.)
- State i's modified divisor went down, and State j's went up
- Thus $\frac{p_i'}{d'} < \frac{p_i}{d}$ and $\frac{p_j'}{d'} > \frac{p_j}{d}$.
- Rearrange: $p_i' < p_i \frac{d'}{d}$ and $p_j' > p_j \frac{d'}{d}$.

Population Monotonicity

Proposition

All divisor methods are population monotone.

Proof.

- Suppose $a_i' < a_i$ and $a_j' > a_j$.
- Want to show: either $p_i' < p_i$ or $p_j' > p_j$.
- We know $p_i' < p_i \frac{d'}{d}$ and $p_j' > p_j \frac{d'}{d}$.
- If $d' < d$ then $\frac{d'}{d} < 1$ so $p_i' < p_i \frac{d'}{d} < p_i$.
- If $d' > d$ then $\frac{d'}{d} > 1$ so $p_j' > p_j \frac{d'}{d} > p_j$.
- Either way, that's what we want to prove.

□

Population Monotonicity and House Monotonicity

Proposition

Any method that is population monotone is also house monotone.

Proof.

- Suppose a method is population monotone
- What happens when house size changes from h to $h+1$?
- Assume no populations change, so $p_i' = p_i$ and $p_j' = p_j$.
- At least one state will gain a seat. We assume $a_j' > a_j$.
- Imagine some other state loses a seat, so $a_i' < a_i$.
- By population monotone, we'd need either $p_i' < p_i$ or $p_j' > p_j$
- But neither thing is true, so that's not possible.

□

Population Monotonicity and Order-Preservingness

Proposition

Any method that is population monotone and neutral must be order-preserving.

Proof.

- Suppose a method is neutral and monotone, and $p_j > p_i$.
- Imagine a new census swapping populations of states i and j
- That is, $p_i' = p_j$ and $p_j' = p_i$, and then $p_k' = p_k$.
- By population monotonicity, $a_i' \geq a_i$ or $a_j' \leq a_j$. (Or both!)
- By neutrality, $a_i' = a_j$ and $a_j' = a_i$.
- Either $a_j = a_i' \geq a_i$, or $a_i = a_j' \leq a_j$.
- Either way, $a_i \leq a_j$, as needed.

□

The New States Paradox

Definition

- Suppose state k is joining the union as a new state, and thus $p_k = 0$ and $p'_k > 0$.
- Suppose there are other states i and j whose populations are unchanged.
- We say a *new states paradox* or *Oklahoma paradox* occurs if $a'_i < a_i$ and $a'_j > a_j$.

- This violates population monotonicity
- No divisor method experiences the new states paradox
- Hamilton's method can experience the new states paradox.

Two Approaches to Population Change

Absolute Change

- How many more or fewer people?
- Gives a number
- $\Delta p_k = p'_k - p_k$

Relative Change

- What is the growth rate?
- Gives a fraction/percentage
- $\frac{\Delta p_k}{p_k} = \frac{p'_k - p_k}{p_k}$

Absolute and Relative Population Change

k	1	2	3
p_k	10,000	10,000	100,000
p'_k	11,000	20,000	110,000
Δp_k	1,000	10,000	10,000
$\Delta p_k / p_k$	0.1	1.0	0.1
%	10%	100%	10%

Discussion Question

- Which state grew the fastest?
- Which state grew the most?

Relative Population Monotonicity

Definition

An apportionment method is relative population monotone if,

- When we consider states with positive population,
- whenever $a_i' < a_i$ and $a_j' > a_j$,
- then $\frac{\Delta p_j}{p_j} > \frac{\Delta p_i}{p_i}$.

Discussion Question

- What does it mean to not have positive population?
- What issue have we talked about that involves non-positive populations?