More on Criteria for Evaluating Divisor Methods

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October 22, 2025

Relative Population Monotonicity

Definition

An apportionment method is relative population monotone if,

- When we consider states with positive population,
- whenever $a_i' < a_i$ and $a_j' > a_j$,
- then $\frac{\Delta p_j}{p_j} > \frac{\Delta p_i}{p_i}$.

Discussion Question

- What does it mean to not have positive population?
- What issue have we talked about that involves non-positive populations?

Relative and Absolute Population Monotonicity

Proposition

If an apportionment method is relative population monotone, then it is population monotone.

Proof.

- Suppose $a_i' < a_i$ and $a_j' > a_j$, and all populations are positive.
- Then $\frac{\Delta p_j}{p_j} > \frac{\Delta p_i}{p_i}$.
- In particular, either $\frac{\Delta p_j}{p_j} > 0$ or $\frac{\Delta p_i}{p_i} < 0$. (Or both!)
- If $\frac{\Delta p_j}{p_j} > 0$ then $\Delta p_j > 0$ so $p_j' > p_j$.
- If $\frac{\Delta p_i}{p_i} < 0$ then $\Delta p_i < 0$ so $p_i' < p_i$.
- Thus the method is (absolute) population monotone.

□

Divisor Methods and Relative Population Monotonicity

Proposition

All divisor methods are relative population monotone.

Proof.

- Suppose $a_i' < a_i$ and $a_j' > a_j$.
- Want to show: $\frac{\Delta p_j}{p_j} > \frac{\Delta p_i}{p_i}$.
- Know that $\frac{p_i'}{d'} < \frac{p_i}{d}$ and $\frac{p_j'}{d'} > \frac{p_j}{d}$.
- Rearranging: $\frac{p_i'}{p_i} < \frac{d'}{d}$ and $\frac{p_j'}{p_j} > \frac{d'}{d}$.
- Combining: $\frac{p_j'}{p_j} > \frac{p_i'}{p_i}$.
- How do we interpret that? State i has grown slower (or shrunk faster) than State j.

Divisor Methods and Relative Population Monotonicity

Proposition

All divisor methods are relative population monotone.

Proof.

- Want to show: $\frac{\Delta p_j}{p_j} > \frac{\Delta p_i}{p_i}$.
- We know: $\frac{p_j'}{p_j} > \frac{p_i'}{p_i}$.
- Subtract 1: $\frac{p_j'}{p_j} - 1 > \frac{p_i'}{p_i} - 1$
- $\frac{p_j' - p_j}{p_j} > \frac{p_i' - p_i}{p_i}$
- That's what we wanted to prove!

Relative and Absolute Population Monotonicity

- If an apportionment method is relative population monotone, then it is population monotone.
- Converse is not true: possible to be absolute population monotone and not relative population monotone
- We say that "relative population monotone" is stronger than "absolute population monotone"
- But it's not very much stronger!

Proposition

If an apportionment method is population monotone and proportional, then it's relative population monotone.

Relative and Absolute Population Monotonicity

Proposition

If an apportionment method is population monotone and proportional, then it's relative population monotone.

Proof.

- What would have to not be relative population monotone?
- Need $a_i' < a_i$ and $a_j' > a_j$
- And also $\frac{\Delta p_j}{p_j} \leq \frac{\Delta p_i}{p_i}$.
- Set $r = 1 + \frac{\Delta p_i}{p_i} = \frac{p_i'}{p_i}$
- Set $s = 1 + \frac{\Delta p_j}{p_j} = \frac{p_j'}{p_j}$.
- $0 < s$ since $0 < p_j'$
- $s \leq r$ by hypothesis.

Relative and Absolute Population Monotonicity

Proposition

If an apportionment method is population monotone and proportional, then it's relative population monotone.

Proof.

- $r = \frac{p_i'}{p_i}$ and $s = \frac{p_j'}{p_j}$
- $0 < s \leq r$.
- Imagine third census scaled up from second: $p_k'' = \frac{p_k'}{r}$.
- By proportionality, $a_k'' = a_k'$.
- $p_i'' = p_i'/r = \frac{p_i'}{p_i'/p_i} = p_i$
- $p_j'' = p_j'/r = \frac{p_j s}{r} \leq p_j$.

Relative and Absolute Population Monotonicity

Proposition

If an apportionment method is population monotone and proportional, then it's relative population monotone.

Proof.

- What do we know?
- $a_i'' = a_i' < a_i$ and $a_j'' = a_j' > a_j$
- $p_i'' = p_i$ and $p_j'' \leq p_j$.
- That violates absolute population monotonicity.

□

- We only want to think about proportional methods
- We can treat absolute and relative population monotonicity as the same.

Criteria Summary

- Hamilton's method:
  - Satisfies quota rule
  - Isn't house monotone
  - Isn't population monotone
- Divisor methods:
  - Are house monotone
  - Aren't population monotone
  - Can they satisfy the quota rule?
  - We know that Jefferson and Adams violate quota
- Can we find a method that satisfies the quota rule, while avoiding the paradoxes of Hamilton's method?

An Impossibility Theorem

Theorem (Balinski and Young)

No apportionment rule that is neutral and population monotone can satisfy the quota rule.

Proof.

- Want to show something *can* happen
- Need a counterexample
- We'll construct a pair of censuses where you cannot be neutral and satisfy the quota rule without violating population monotonicity.

An Impossibility Theorem

Theorem (Balinski and Young)

No apportionment rule that is neutral and population monotone can satisfy the quota rule.

Proof.

- Allocate ten seats to the following two censuses:

$$p_1 = 69,900 \qquad p'_1 = 68,000$$
$$p_2 = 5,200 \qquad p'_2 = 5,500$$
$$p_3 = 5,000 \qquad p'_3 = 5,600$$
$$p_4 = 19,900 \qquad p'_4 = 5,700.$$

An Impossibility Theorem

Theorem (Balinski and Young)

No apportionment rule that is neutral and population monotone can satisfy the quota rule.

Proof.

k	p_k	q_k	$\lfloor q_k \rfloor$	$\lceil q_k \rceil$
1	69,900	6.99	6	7
2	5,200	0.52	0	1
3	5,000	0.50	0	1
4	19,900	1.99	1	2

- State 1 gets at most 7
- State 4 gets at most 2
- Either State 2 or State 3 has to get one
- By order-preserving, State 2 has to get at least one seat.

An Impossibility Theorem

Theorem (Balinski and Young)

No apportionment rule that is neutral and population monotone can satisfy the quota rule.

Proof.

k	p_k	q_k	$\lfloor q_k \rfloor$	$\lceil q_k \rceil$
1	68,000	8.02	8	9
2	5,500	0.65	0	1
3	5,600	0.66	0	1
4	5,700	0.67	0	1

- State 1 gets at least 8
- At most two of the other states get seats
- By order preserving, State 2 can't get a seat.

An Impossibility Theorem

Theorem (Balinski and Young)

No apportionment rule that is neutral and population monotone can satisfy the quota rule.

Proof.

$$\begin{aligned} p_1 &= 69,900 & p_1' &= 68,000 \\ p_2 &= 5,200 & p_2' &= 5,500 \\ p_3 &= 5,000 & p_3' &= 5,600 \\ p_4 &= 19,900 & p_4' &= 5,700. \end{aligned}$$

- $a_1 \leq 7$ and $a_1' \geq 8$
- $a_2 = 1$ but $a_2' = 0$
- But $p_1 > p_1'$ and $p_2 < p_2'$
- Violates population monotonicity.

□

An Impossibility Theorem

Theorem (Balinski and Young)

No apportionment rule that is neutral and population monotone can satisfy the quota rule.

Corollary

No divisor method can satisfy the quota rule.

- Can't satisfy quota and population monotone
- But what about house monotone?