Zero-Sum Games

Jay Daigle

jaydaigle@gwu.edu

https://jaydaigle.net/politics

The George Washington University

October 27, 2025

Battle of the Bismarck Sea

- In WWII, the Japanese Navy needed to resupply New Guinea
- Had to sail around New Britain, either North or South path
- US controlled New Britain, wanted to bomb Japanese Convoy

US Strategy

- US can focus its search to the North or the South
- If the Japanese go North:
  - Guess North: 2 days of bombing
  - Guess South: 1 day of bombing
- If the Japanese go South:
  - Guess North: 2 days of bombing
  - Guess South: 3 days of bombing
- What should the US do?

Conflict and Game Theory

- Game theory models strategic interactions
- How do you make choices when other people with different goals are also making choices?
- Developed in early 20th century
- Reached prominence after World War II:
  - Avoid another world war
  - Avoid nuclear war
- Summarize strategic situation in a mathematical model
- Find optimal strategy or strategies
- Can we find ways to cooperate?

Zero-Sum Games

Definition

- A two-person zero-sum game is a game featuring two players, in which
  - Each player adopts a strategy
  - The combination of strategies determines a number called the payoff.
- We can think of the payoff as the amount of money player 2 has to pay player 1
- If they payoff is negative, this means player 1 has to pay player 2.

Zero-Sum Games

- Can represent a two-player zero-sum game with a matrix, which is a grid of possible payoffs
- We call player 1 Row and player 2 Column
- If Row has m choices and Column has n choices, we get a $m \times n$ matrix
- The entry on row i, column j is $u_{i,j}$

	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$u_{1,1}$	$u_{1,2}$	$u_{1,3}$
$i = 2$	$u_{2,1}$	$u_{2,2}$	$u_{2,3}$
$i = 3$	$u_{3,1}$	$u_{3,2}$	$u_{3,3}$

Battle of the Bismarck Sea

- Two players: US and Japan
- Zero-sum: US wants maximum bombing and Japan wants minimum

		Japan (Column)	
		North	South
United States (Row)	North	2	2
	South	1	3

Roshambo

Example (Roshambo, or Rock Paper Scissors)

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

- Positive numbers are good for player 1 Row
- Negative numbers are good for player 2 Column

Naive Strategies

Definition

- The outcome that gives a player their best possible payoff is the primary outcome.
- For Row this is the largest entry in the payoff matrix
- For Column it is the smallest, or most negative, entry.

Definition

- In the naive method, a player chooses the strategy corresponding to their primary outcome.
- This is called the player's naive strategy, or sometimes the greedy strategy or optimistic strategy.
- If both players play their naive strategies, we get the doubly naive outcome.

Naive Strategies

Example (The Battle of the Bismarck Sea)

		Japan (Column) North	South
United States (Row)	North	2	2
	South	1	3

Primary outcome for USA

- 3 days of bombing
- Searching to the South

Primary outcome for Japan

- 1 day of bombing
- Going to the North

- Doubly naive outcome: one day of bombing

Naive Strategies

Example (Rock Paper Scissors)

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

Primary outcome for Row

- Win, 1 point
- Any strategy

Primary outcome for Column

- Loss, -1 point
- Any strategy

- Doubly naive outcome: Any??

Prudent Strategies

Definition

- The worst payoff a player can get from a given strategy is that strategy's guarantee.
- (The player is guaranteed to get at least that good a payoff.)

Definition

- In the prudent method, a player chooses the strategy with the best guarantee, called the prudent strategy.
- We might also call this the pessimistic strategy, because it wants to minimize the damage from the worst-case scenario.
- If both players play their prudent strategies, we get the doubly prudent outcome.

Prudent Strategies

Example (The Battle of the Bismarck Sea Min-Max Diagram)

		Japan (Column) North	Japan (Column) South	
United States (Row)	North	2	2	2
	South	1	3	1
		2	3	

Prudent strategy for USA

- Search North

Prudent strategy for Japan

- Go to the North

- Doubly prudent outcome: two days of bombing

Naive and Prudent Strategies

Naive strategy

- Maximax: maximize your maximum payoff.
- "Hope chess": hope your opponent plays into your hands
- Dangerous if your opponent is smart!

Prudent strategy

- Minimax: minimize your maximum loss
- maximin: Maximize your minimum payoff
- Non-obvious mathematical conclusion: those two ideas are the same.

Prudent strategies

Proposition

- *Let r be the guarantee of Row's prudent strategy, and c be the guarantee of Column's prudent strategy.*
- *If Row plays a prudent strategy, their payoff will be at least r*
- *But if Row plays a non-prudent strategy, it is possible their payoff will be lower than r.*
- *Similarly, if Column plays a prudent strategy, their payoff will be at most c*
- *But if Column plays a non-prudent strategy, it is possible their payoff will be greater than c.*
- *(Remember Column prefers smaller payoffs)*

Prudent strategies

Proposition

- *If Row plays a prudent strategy their payoff will be at least r*
- *If Row plays a prudent strategy their payoff will be at most c*

Proof.

- Suppose row i and column j are prudent strategies
- The payoff to this pair of strategies is $u_{i,j}$
- Since i is prudent, we must have $u_{i,j} \geq r$
- Since j is prudent, we must have $u_{i,j} \leq c$

$\square$

Prudent strategies

Proposition

- *If Row plays a prudent strategy their payoff will be at least r*
- *If Row plays a prudent strategy their payoff will be at most c*

Corollary

If r is the guarantee of Row's prudent strategy and c is the guarantee of Column's prudent strategy, then $r \leq c$.

Proof.

- If row i and column j are prudent, then
  - $u_{i,j} \geq r$
  - $u_{i,j} \leq c$.
- Thus $r \leq u_{i,j} \leq c$.

□

Best Response

Discussion Question

If you know what your opponent will do, how should that affect your choices?

Definition

- A strategy choice by one player is called the best response to an opponent strategy if it gives the best payoff against that strategy.
- The best response to a naive strategy is called the counter-naive strategy.
- The best response to a prudent strategy is called the counter-prudent strategy.

Best Response

Example

		Japan (Column)	
		North	South
United States (Row)	North	2	2
	South	1	3

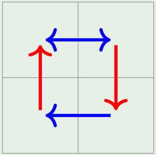
- If Japan goes North, US should search North
- If Japan goes South, US should search South
- If US searches South, Japan should go North
- If US searches North, Japan should . . . ?

Best Response and Flow Diagrams

- Summarize our best responses with a flow diagram
- A map that gives us the best response to any strategy
- Each column: vertical arrow points to *largest* entry (or entries)
- Each row: horizontal arrow points to *smallest* entry

Example (Battle of the Bismarck Sea)

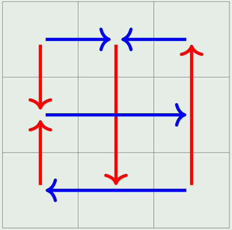
	North	South
North	2	2
South	1	3



Best Response and Flow Diagrams

Example (Rock Paper Scissors)

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0



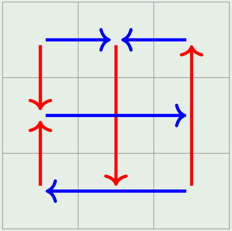
Backward Induction

- If I know what you'll do, I can choose the best response
- But if you know that, you can choose the best response to my choice
- But if I know that, I can choose the best response to that choice
- But if you know that, you can . . .
- We call this reasoning process backward induction
- Where does it stop?

Backward Induction: Rock Paper Scissors

Example (Rock Paper Scissors)

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

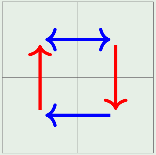


- If Row plays Rock, then Column should play Paper
- But if Column plays Paper, then Row should play Scissors
- But if Row plays Scissors, then Column should play Rock
- But if . . .

Backward Induction: Bismarck Sea

Example (Battle of the Bismarck Sea)

	North	South
North	2	2
South	1	3



- If Japan goes South, then the US should go South
- But if the US goes South, then Japan should go North
- But if Japan goes North, the US should go North
- But if the US goes North . . .
- Japan is fine sticking with North!

Saddle Points

Definition

- A saddle point is an outcome such that the strategy for each player is the best response to the strategy of the opponent, simultaneously.
- A saddle point strategy is a strategy that corresponds to a saddle point outcome.

- An outcome is a saddle point if and only if all the arrows in its row and column point to it.
- The point in row k, column ℓ is a saddle point if

$$u_{k,\ell} \leq u_{k,j} \qquad \text{for any column } j$$

$$u_{k,\ell} \geq u_{i,\ell} \qquad \text{for any row } i.$$

Saddle Points

Theorem

- *A two-person zero-sum game has a saddle point if and only if* $r = c$.
- *In that case, the saddle point is a doubly prudent outcome, and the payoff is* $r = c$.