Strategies, Outcomes, and Saddle Points

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Zero-Sum Games

- Can represent a two-player zero-sum game with a matrix
- We call player 1 Row and player 2 Column
- Row has m choices, Column has n choices, get a $m \times n$ matrix
- The entry on row i, column j is $u_{i,j}$

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
$i = 1$	$u_{1,1}$	$u_{1,2}$	$u_{1,3}$	$u_{1,4}$
$i = 2$	$u_{2,1}$	$u_{2,2}$	$u_{2,3}$	$u_{2,4}$
$i = 3$	$u_{3,1}$	$u_{3,2}$	$u_{3,3}$	$u_{3,4}$

Zero-Sum Games

	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	$u_{1,1}$	$u_{1,2}$	$u_{1,3}$	$u_{1,4}$
$i=2$	$u_{2,1}$	$u_{2,2}$	$u_{2,3}$	$u_{2,4}$
$i=3$	$u_{3,1}$	$u_{3,2}$	$u_{3,3}$	$u_{3,4}$

- A row i or column j is a strategy
- The square (i, j) is an outcome
- The number $u_{i,j}$ is the payoff
- Positive numbers are good for Row
- Negative numbers are good for Column

Battle of the Bismarck Sea

- Two players: US and Japan
- Zero-sum: US wants maximum bombing and Japan wants minimum

		Japan (Column)	
		North	South
United States (Row)	North	2	2
	South	1	3

Roshambo

Example (Roshambo, or Rock Paper Scissors)

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

Naive Strategies

Definition

- The primary outcome give a player their best payoff.
- Row: largest entry in the matrix
- Column: smallest, or most negative, entry.

Definition

- The naive strategy is the strategy that can produce the primary outcome
- Also the greedy strategy or optimistic strategy or maximax.
- The naive method: choose the strategy corresponding to your primary outcome.
- If both players play their naive strategies, we get the doubly naive outcome.

Naive Strategies

Example (The Battle of the Bismarck Sea)

Japan (Column)

		North	South
United States (Row)	North	2	2
	South	1	3

USA

- Primary *outcome*: S, S
- Payout: 3
- Naive strategy: South

Japan

- Primary outcome: S, N
- Payout: 3
- Naive strategy: North

- Doubly naive outcome: S,N. One day of bombing

Naive Strategies

Example (Rock Paper Scissors)

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

Primary outcome for Row

- Primary outcome: RS, PR, SP
- Payoff: $1 = u_{1,3} = u_{2,1} = u_{3,2}$
- Naive strategy: Any strategy

Prudent Strategies

Definition

- The worst payoff for a strategy k its guarantee.
- If you play k you're guaranteed to get at least this much.
- Guarantee for row k: $\min u_{k,j}$ for all columns j.
- Guarantee for column ℓ: $\max u_{i,\ell}$ for all rows i.

Definition

- Prudent strategy: strategy with the best guarantee
- Also the pessimistic strategy, minimax, maximin
- prudent method: choose the prudent strategy.
- If both players play their prudent strategies, we get the doubly prudent outcome.

Prudent Strategies

Example (The Battle of the Bismarck Sea Min-Max Diagram)

Japan (Column)

United States (Row)		North	South	
	North	2	2	2
	South	1	3	1
		2	3	

Prudent strategy for USA

- Search North

Prudent strategy for Japan

- Go to the North

- Doubly prudent outcome: North, north; two days of bombing

Prudent Strategies

Proposition

- *Let r be the guarantee of Row's prudent strategy, and c be the guarantee of Column's prudent strategy.*
- *If Row plays a prudent strategy, their payoff will be at least r*
- *Algebraically: If k is prudent, then $u_{k,j} \geq r$ for all j*
- *Similarly, if Column plays a prudent strategy, their payoff will be at most c*
- *If ℓ is prudent, then $u_{i,\ell} \leq c$ for all i.*

Prudent Strategies

Proposition

- *If Row plays a prudent strategy their payoff will be at least r*
- *If Column plays prudent strategy their payoff will be at most c*

Corollary

If r is the guarantee of Row's prudent strategy and c is the guarantee of Column's prudent strategy, then $r \leq c$.

Proof.

- Suppose row k and column ℓ are prudent strategies
- Since k is prudent, we must have $u_{k,\ell} \geq r$
- Since ℓ is prudent, we must have $u_{k,\ell} \leq c$
- Thus $r \leq u_{k,\ell} \leq c$.

□

Best Response

Definition

- The best response to an fixed opponent strategy is the strategy that gives the best payoff against the opponent strategy.
- Row k is the best response to column ℓ if $u_{k,\ell} \geq u_{i,\ell}$ for all i.
- Column ℓ is the best response to row k if $u_{k,\ell} \leq u_{k,j}$ for all j.

Definition

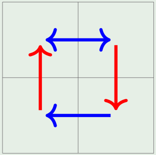
- Counter-naive strategy: best response to the naive strategy
- Counter-prudent strategy: best response to the prudent strategy

Best Response and Flow Diagrams

- Summarize our best responses with a flow diagram
- A map that gives us the best response to any strategy
- Each column: vertical arrow points to *largest* entry (or entries)
- Each row: horizontal arrow points to *smallest* entry

Example (Battle of the Bismarck Sea)

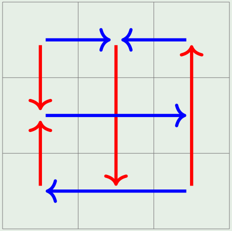
	North	South
North	2	2
South	1	3



Backward Induction: Rock Paper Scissors

Example (Rock Paper Scissors)

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

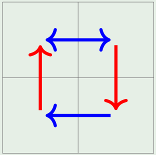


- If Row plays Rock, then Column should play Paper
- But if Column plays Paper, then Row should play Scissors
- But if Row plays Scissors, then Column should play Rock
- But if . . .

Backward Induction: Bismarck Sea

Example (Battle of the Bismarck Sea)

	North	South
North	2	2
South	1	3



- If Japan goes South, then the US should go South
- But if the US goes South, then Japan should go North
- But if Japan goes North, the US should go North
- But if the US goes North . . .
- Japan is fine sticking with North!

Saddle Points

Definition

- Saddle point: an outcome where each player's strategy is a best response to the opponent's strategy, simultaneously.

- (k, ℓ) is a saddle point if:
  - Row k is a best response to column ℓ
    - That means $u_{k,\ell} \geq u_{i,\ell}$ for any row i.
  - And column ℓ is a best response to row k.
    - That means $u_{k,\ell} \leq u_{k,j}$ for any column j
- An outcome is a saddle point if and only if all the arrows in its flow diagram in its row and column point in towards it.

Definition

Saddle point strategy: strategy that corresponds to a saddle point outcome.

Saddle Points

Theorem

- *A 2P zero-sum game has a saddle point if and only if $r = c$.*
- *The saddle point is doubly prudent, with payoff $r = c$.*

Proof.

- "If and only if": need to prove two things.
- Assume $r = c$, and prove that the game has a saddle point
- Assume the game has a saddle point, and prove $r = c$.

Saddle Points

Theorem

- *A 2P zero-sum game has a saddle point if and only if* $r = c$.
- *The saddle point is doubly prudent, with payoff* $r = c$.

Proof.

- Suppose $r = c$.
- Want to prove there's a saddle point.
- Let row k have guarantee r.
- Then $u_{k,j} \geq r$ for all j.
- Let column ℓ have guarantee $c = r$
- Then $u_{i,\ell} \leq r$ for all i.
- In particular, $u_{k,\ell} = r$.

Saddle Points

Theorem

- *A 2P zero-sum game has a saddle point if and only if $r = c$.*
- *The saddle point is doubly prudent, with payoff $r = c$.*

Proof.

- Assuming $r = c$
- $u_{k,\ell} = r = c$.
- $u_{k,j} \geq r$ for all j, and $u_{k,\ell} = r$; so ℓ is a best response to k.
- $u_{i,\ell} \leq r$ for all i, and $u_{k,\ell} = r$; so k is a best response to ℓ.
- Thus (k, ℓ) is a saddle point.

Saddle Points

Theorem

- *A 2P zero-sum game has a saddle point if and only if $r = c$.*
- *The saddle point is doubly prudent, with payoff $r = c$.*

Proof.

- Conversely, suppose (k, ℓ) is a saddle point
- $u_{k,\ell}$ must be the smallest entry in row k,
  - If $u_{k,j}$ is smaller, then j would be a better response.
- Thus $u_{k,\ell}$ is the guarantee of row k
- But r is the largest guarantee of any row, so $u_{k,\ell} \leq r$.

Saddle Points

Theorem

- *A 2P zero-sum game has a saddle point if and only if $r = c$.*
- *The saddle point is doubly prudent, with payoff $r = c$.*

Proof.

- Suppose (k, ℓ) is a saddle point
- $u_{k,\ell}$ must be the largest entry in column ℓ
- Thus $u_{k,\ell}$ is the guarantee of column ℓ
- But $r = c$ is the smallest guarantee of any column, so $u_{k,\ell} \geq r$.

Saddle Points

Theorem

- *A 2P zero-sum game has a saddle point if and only if $r = c$.*
- *The saddle point is doubly prudent, with payoff $r = c$.*

Proof.

- Suppose (k, ℓ) is a saddle point
- Because $u_{k,\ell}$ is the smallest entry in row k, saw that $u_{k,\ell} \leq r$
- Because $u_{k,\ell}$ is the largest entry in column ℓ, saw that $u_{k,\ell} \geq r$.
- Can only both be true if $r = u_{k,\ell}$.

Saddle Points

Theorem

- *A 2P zero-sum game has a saddle point if and only if $r = c$.*
- *The saddle point is doubly prudent, with payoff $r = c$.*

Corollary

An outcome (k, ℓ) is a saddle point if and only if row k is a saddle point strategy and column ℓ is a saddle point strategy.

Discussion Question

- Isn't that obvious?
- What is this corollary trying to say?

Saddle Points

Theorem

- *A 2P zero-sum game has a saddle point if and only if $r = c$.*
- *The saddle point is doubly prudent, with payoff $r = c$.*

Corollary

An outcome (k, ℓ) is a saddle point if and only if row k is a saddle point strategy and column ℓ is a saddle point strategy.

- Obvious: if (k, ℓ) is a saddle point then k and ℓ are saddle point strategies.
- But what if (k, j) and (i, ℓ) are saddle points?
- Then k and ℓ are both saddle point strategies
- Does (k, ℓ) also have to be a saddle point?

Saddle Points

Corollary

An outcome (k, ℓ) is a saddle point if and only if row k is a saddle point strategy and column ℓ is a saddle point strategy.

Proof.

- Suppose k and ℓ are saddle point strategies.
- There are saddle points (k, j) and (i, ℓ) for some column j and row i
- Claim (k, ℓ) is a saddle point.
- By theorem, $r = c$ and k and ℓ are prudent strategies.
- That means (k, ℓ) is doubly prudent. By theorem, it's a saddle point.

□

Saddle Points and Min-Max Diagrams

- Idea: we can find saddle points with a min-max diagram.

2	3	2	2	5	2
0	10	0	3	-9	-9
-2	2	-1	2	7	-2
2	10	2	2	2	2
0	4	1	0	-4	-4
2	10	2	3	7	

- $r = 2$
- $c = 2 = r$
- Saddle points: (1,1), (1,3), (4,1), (4,4)

Saddle Points

Theorem

- *A 2P zero-sum game has a saddle point if and only if $r = c$.*
- *The saddle point is doubly prudent, with payoff $r = c$.*

Corollary

A strategy in a two-person zero-sum game is a saddle point strategy if and only if it is both prudent and counter-prudent.

Proof.

- A saddle point strategy must be prudent
- The other player's saddle point strategy is also prudent
- So a saddle point strategy must be a best response to a prudent strategy.

□

Dominant Strategies

- Idea: instead of looking for good strategies, look for bad ones

Definition

- One row of a matrix dominates another if:
  - each entry of the first row is at least as large as the corresponding entry of the second row, and
  - at least one entry is strictly larger.
- Algebraically: row k dominates row i if:
  - $u_{k,j} \geq u_{i,j}$ for each column j, and
  - There is at least one column j such that $u_{k,j} > u_{i,j}$.
- We say row k strictly dominates row i if $u_{k,j} > u_{i,j}$ for each column j.

- If k dominates i then Row should never play i.

Dominant Strategies

Example

1	2	4
5	3	6

- Row 2 dominates Row 1.
- Row 2 *strictly* dominates Row 1.

Dominant Strategies

Example

1	2	3
5	1	6

- Neither row dominates the other since $u_{1,1} < u_{1,2}$ but $u_{2,1} > u_{2,2}$.

Example

1	2	3
1	2	3

- Neither row dominates the other since they're identical.

Dominant Strategies

Definition

- One column of a matrix dominates another if:
  - each entry of the first column is at least as small as the corresponding entry of the second column, and
  - at least one entry is strictly smaller.
- Algebraically: column ℓ dominates column j if:
  - $u_{i,\ell} \leq u_{i,j}$ for each row i, and
  - There is at least one row i such that $u_{i,\ell} < u_{i,j}$.
- We say column ℓ strictly dominates column j if $u_{i,\ell} < u_{i,j}$ for each row i.

- If ℓ dominates j then Column should never play j.

Dominant Strategies

Example

1	2	4
5	3	6

- Column 1 (strictly) dominates Column 3
- Column 2 (strictly) dominates Column 3
- Neither Column 1 nor Column 2 dominates the other.

- A reasonable player might pick Column 1 or Column 2,
- but they'd never pick Column 3.

Dominant Strategies and Reduction

- Idea: can ignore dominated strategies
- This won't change the analysis of the game
- Eliminated dominated strategies is reduction
- Sometimes a reduction step causes more strategies to become dominated
- When we can't reduce any further, we have a complete reduction

Dominant Strategies and Reduction

Example

US \ Japan	North	South
North	2	2
South	1	3

- C1 dominates C2

US \ Japan	North
North	2
South	1

- Now R2 dominates R1.

US \ Japan	North
North	2

- Same conclusion we reached with the flow diagram.

Example

0	-1	-2	5	4
-3	1	2	3	6
-4	-5	-6	-7	7

Naive strategies

- Row: 3, aiming for $u_{3,5} = 7$
- Column: 4, aiming for $u_{3,4} = -7$
- Doubly naive: $u_{3,4} = -7$.

Counter-naive

- Row: 1 gets $u_{1,4} = 5$
- Column: 4 gets $u_{3,4} = -7$

Example

0	-1	-2	5	4
-3	1	2	3	6
-4	-5	-6	-7	7

Prudent strategies

- Row: guarantees $-2, -3, -7$
- Prudent: R1
- Column: guarantees $0, 1, 2, 5, 7$
- Prudent: C1
- Doubly prudent: $u_{1,1} = 0$

Counter-prudent strategies

- Row: 1, gets $u_{1,1} = 0$
- Column: 3, gets $u_{1,3} = -2$

Counter-counter-prudent

- Row: 2 gets $u_{2,3} = 2$
- Column: 3 gets $u_{1,3} = -2.$

Example

0	−1	−2	5	4
−3	1	2	3	6
−4	−5	−6	−7	7

- Saddle points? Could draw full flow diagram
- But first look for dominated strategies.
- C5 dominated by C1, C2, C3

0	−1	−2	5
−3	1	2	3
−4	−5	−6	−7

- Row's original primary strategy is gone!
- (This is why the naive strategy is naive.)

Example

0	-1	-2	5	4
-3	1	2	3	6
-4	-5	-6	-7	7

- R3 dominated by R1 or R2

0	-1	-2	5
-3	1	2	3

0	-1	-2	5
-3	1	2	3
-4	-5	-6	-7

- C4 dominated by C1, C2, C2

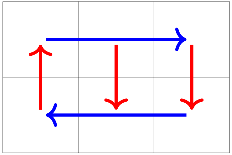
0	-1	-2
-3	1	2

Example

0	−1	−2	5	4
−3	1	2	3	6
−4	−5	−6	−7	7

$\rightarrow$

0	−1	−2
−3	1	2



- This has no saddle point

- Thus original game also has no saddle point
- Reasonable Row players: R1 or R1
- Reasonable Column players: C1, C2, C3

The Dilemma

- If there's a saddle point, we should pick that
- If there's no saddle point, there's no clear best choice
- But if we settle on one strategy, we can be exploited.

Discussion Question

How do we pick a strategy?